Structural GARCH: The Volatility-Leverage Connection*

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Abstract

We propose a new model of volatility where financial leverage amplifies equity volatility by what we call the "leverage multiplier". The exact specification is motivated by standard structural models of credit; however, our parametrization departs from the classic Merton (1974) model and is, as we show, flexible and accurate enough to capture environments where the firm's asset volatility is stochastic, asset returns can jump, and asset shocks are non-normal. As a result, our model also provides estimates of daily asset returns and asset volatility. In addition, our specification nests both a standard GARCH and the Merton model, which allows for a simple statistical test of how leverage interacts with equity volatility. Empirically, the Structural GARCH model outperforms a standard GARCH model for approximately 75% of the financial firms we analyze. We then apply the Structural GARCH model to two empirical applications: the leverage effect and systemic risk measurement.

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1 Introduction

In the financial crisis of 2007-2009 it is widely observed that equity volatilities hit sustained levels not seen since the Great Depression. This was particularly true for the financial sector. Yet it is also observed that the leverage in the financial sector reached extremely high levels. When firm leverage, measured by firm asset value divided by equity value, is high it is not surprising that equities would be highly volatile. Is it possible that asset volatility did not rise during the financial crisis even though equity volatility hit such high levels?

In this paper we propose an econometric approach to disentangle the effects of leverage on equity volatility. The model is based on the classic Merton (1974) model of firm capital structure, extended to incorporate time variation in asset volatility. A non-exhaustive list of theoretical extensions of the Merton (1974) model include Black and Cox (1976), Leland and Toft (1996), Collin-Dufresne and Goldstein (2001), and McQuade (2013). While these studies provide valuable theoretical insights regarding capital structure and credit spreads, our goal is empirical in nature. We aim to develop a flexible econometric specification to capture the relationship between leverage and volatility. Our approach begins by examining the theoretical relationship between asset volatility and equity volatility through the lens of structural models of credit. Through this exposition, we introduce the "leverage multiplier", which quantifies how leverage amplifies asset volatility into equity volatility.[1] We explore the functional form of the leverage multiplier in a variety of settings and show that even when assets follow stochastic volatility and experience jumps, equity volatility is well-captured by our leverage multiplier. Furthermore, we show that the leverage multiplier takes roughly the same form across different asset return specifications.

Under reasonable parameterizations, the volatility of asset volatility and jumps in assets do indeed matter for the level of equity. However, the volatility of asset volatility and jumps mean much less for high frequency equity returns and volatility. The intuition is simple: the time it takes volatility to mean revert is much shorter than typical debt maturities, and thus long run forecasts of cumulative asset volatility are virtually constant. In turn, daily equity returns are dictated primarily by leverage and daily asset returns. Similarly, daily equity volatility is primarily determined by leverage and the level of asset volatility. These findings are echoed by the Schaefer and Strebulaev (2008) finding that hedge ratios from the Merton (1974) model may be quite accurate even though the levels are not well determined.

[1] Another important feature of this model is non-linear amplification by leverage.

Our theoretical exploration of the properties of the leverage multiplier leads to a simple econometric specification of equity and asset returns. We call this specification a Structural GARCH model given its theoretical underpinnings. It takes advantage of simple mathematical transformations of the functions delivered by the Black-Scholes-Merton model, but is designed to provide the flexibility to capture a number of stochastic asset volatility (and jump) settings. One of the key features of our model is that higher levels of leverage result in higher amplification of asset shocks on equity returns, as well as higher (stochastic) equity volatility. Because this was apparent in the recent financial crisis, we estimate our model for a number of financial firms and our subsequent empirical analysis focuses on firms in this sector.

Our empirical results show that incorporating leverage via the leverage multiplier as in our Structural GARCH model outperforms a simple vanilla asymmetric GARCH model of equity returns. Our model nests a vanilla GARCH, and thus provides a natural way to access the statistical significance of leverage for equity volatility.[2] For the sample of firms we examine, nearly 75% favor our Structural GARCH model over a plain asymmetric GARCH model. Since the Structural GARCH model delivers a daily series for asset volatility, we are also able to study the joint dynamics of asset volatility and leverage in the build up to the financial crisis. The empirical results reveal that at the onset of the financial crisis, the rise in equity volatility was primarily due to rising leverage, but later phases also include substantial rises in asset volatility.

We use our Structural GARCH model in two applications: determining the sources of asymmetric equity volatility and measuring systemic risk. Equity volatility asymmetry refers to the well-known negative correlation between equity returns and equity volatility. One popular explanation for this empirical regularity is leverage. Namely, when a firm experiences negative equity returns, its leverage mechanically rises, and thus the firm has more risk (volatility). Some examples of previous work on this topic include Black (1976), Christie (1982), Golsten et al. (1993), and Beakart and Wu (2001). The challenge faced by previous studies is that asset returns are unobservable, so teasing out the causes of volatility asymmetry requires alternative strategies. To this end, the Structural GARCH model provides a simple way to explore this hypothesis, given that we obtain high frequency asymmetric GARCH parameter estimates for assets, after controlling for leverage.[3] Following intuition, we find that, on average, firms with more leverage exhibit a bigger gap

[2] Indeed, asymmetric GARCH models have been interpreted as capturing the interaction between leverage and equity volatility. This is famously known as the "leverage effect" of Black (1976) and Christie (1982). In contrast, our model directly incorporates volatility asymmetry at the asset level and also directly incorporates leverage into equity volatility. As we will discuss shortly, this also allows us to tease out the root of the observed leverage effect.

[3] Also in contrast to previous work, our model directly incorporates risky debt into the equity return specification. The emphasis of risky debt is important, since it introduces important non-linear interactions between leverage and equity volatility.

between the asymmetry of their equity return volatility and their asset return volatility. Nonetheless, we find that the overall contribution of measured leverage to the so-called "leverage effect" is somewhat weak; for our sample of firms, leverage accounts for only about 23% of equity volatility asymmetry.[4]

The second application of our Structural GARCH model involves systemic risk measurement. We extend the SRISK measure of Acharya et al. (2012) and Brownlees and Engle (2012) by incorporating the Structural GARCH model for firm level equity returns. The SRISK measure captures how much capital a firm would need in the event of another financial crisis, where a financial crisis is proxied by a decline of 40% over 6 months in the aggregate stock market. Importantly, the leverage amplification mechanism that is built into our model naturally embeds the types of volatility-leverage spirals observed during the crisis. It is precisely this feature that makes leverage an important consideration even in time of low volatility since a negative sequence of equity returns increases leverage, and further amplifies negative shocks to assets. Accordingly, we show that using the Structural GARCH model for systemic risk measurement shows promise in providing earlier signals of financial firm distress. Compared with models that do not incorporate leverage amplifications explicitly, the model-implied expected capital shortfall in a crisis for firms such as Citibank and Bank of America rises much earlier prior to the financial crisis (and remains as high or higher through the crisis). Thus, Structural GARCH serves as an important step towards developing countercyclical measures of systemic risk that may also motivate policies which prevent excess leverage from building within the financial system.

Section 2 introduces the Structural GARCH model and its economic underpinnings. Here, we will use basic ideas from structural models of credit to explore the relationship between leverage and equity volatility, which leads to a natural econometric specification for equity and asset returns. In Section 3, we describe the data we use in our empirical work, along with some technical issues regarding estimation of the model. Section 4 describes our empirical results and explores some aggregate implications of our model. In Section 5, we apply the Structural GARCH model to two applications: asymmetric volatility in equity returns, and systemic risk measurement. Finally, Section 6 concludes and suggests more applications of the Structural GARCH model.

[4] These results are consistent with Beakart and Wu (2001) and Lo and Hasanhodzic (2011) who find that leverage does not appear to fully explain the asymmetry in equity volatility. Beakart and Wu (2001), however, assume that debt is riskless and therefore is silent about the nonlinear interaction between equity volatility and leverage. Lo and Hasanhodzic (2011) focus on a subset of firms with no leverage, which we do not pursue in this paper.

2 Structural GARCH

Our goal is to explore the relationship between the leverage of a firm and its equity volatility. A simple framework to explore this relationship is the classical Merton (1974) model of credit risk and extensions of this seminal work. Equity holders are entitled to the assets of the firm that exceed the outstanding debt. As Merton observed, equity can then be viewed as a call option on the total assets of a firm with the strike of the option being the debt level of the firm. In this model, the fact that firms have outstanding debt of varying maturities is ignored, and we will also adopt this assumption for the sake of maintaining a simple econometric model. The purpose of using structural models is to provide economic intuition for how leverage and equity volatility should interact. It is worth emphasizing that we call our volatility model a "Structural GARCH" because it is motivated from this analysis, but not because it derives precisely from a particular option pricing model. In fact, one advantage of our approach is our ability to remain relatively agnostic about the true option pricing model that underlies the data generating process.

2.1 Motivating the Model

As is standard in structural models of credit, the equity value of a firm is a function of the asset process and the debt level of the firm. We can therefore define the equity value as follows:

$$E_t = f\left(A_t, D_t, \sigma_{A,t}, \tau, r_t\right) \tag{1}$$

where $f(\cdot)$ is an unspecified call option function, A_t is the current market value of assets, D_t is the current book value of outstanding debt, $\sigma_{A,t}$ is the (potentially stochastic) volatility of the assets. τ is the life of the debt, and finally, r_t is the annualized risk-free rate at time t.[5] In order for the relationship in Equation (1) to hold true in a general sense, we assume that the future distribution of volatility is unimportant for the option value. As we will see, our volatility model is flexible enough that this is not a restrictive assumption.

[5] In Appendix A, we re-derive all of the subsequent results in the presence of asset jumps. Since we find the results to be essentially unchanged, we focus on the simpler case of only stochastic volatility for the sake of brevity.

Next, we specify the following generic process for assets and variance:

$$\begin{aligned}\frac{dA_t}{A_t} &= \mu_A(t)dt + \sigma_{A,t}dB_A(t) \\ d\sigma_{A,t}^2 &= \mu_v(t,\sigma_{A,t})dt + \sigma_v(t,\sigma_{A,t})dB_v(t)\end{aligned} \tag{2}$$

where $dB_A(t)$ is a standard Brownian motion. $\sigma_{A,t}$ captures potential time-varying asset volatility, which we will model formally in Section 2.4.[6] The process we specify for asset volatility is general enough to capture popular stochastic volatility models such as Heston (1993) or the Ornstein-Uhlenbeck process employed by, for example, Stein and Stein (1991). We allow an arbitrary instantaneous correlation of ρ_t between the shock to asset returns, $dB_A(t)$, and the shock to asset volatility, $dB_v(t)$. The specification in Equation (2) encompasses a wide range of stochastic volatility models that are popular in the option pricing literature.[7]

The instantaneous return on equity is computed via simple application of Itō's Lemma:

$$\frac{dE_t}{E_t} = \Delta_t \frac{A_t}{D_t}\frac{D_t}{E_t}\cdot\frac{dA_t}{A_t} + \frac{\nu_t}{E_t}\cdot d\sigma_{A,t} + \frac{1}{2E_t}\left[\frac{\partial^2 f}{\partial A_t^2}d\left\langle A\right\rangle_t + \frac{\partial^2 f}{d\left(\sigma_{A,t}\right)^2}d\left\langle \sigma_A^f\right\rangle_t + \frac{\partial^2 f}{\partial A \partial \sigma_{A,t}}d\left\langle A,\sigma_A\right\rangle_t\right] \tag{3}$$

where $\Delta_t = \partial f\left(A_t, D_t, \sigma_{A,t}, \tau, r\right)/\partial A_t$ is the "delta" in option pricing, $\nu_t = \partial f\left(A_t, D_t, \sigma_{A,t}, \tau, r\right)/\partial \sigma_{A,t}$ is the so-called "vega" of the option, and $\left\langle X\right\rangle_t$ denotes the quadratic variation process for an arbitrary stochastic process X_t. Here, we have ignored the sensitivity of the option value to the maturity of the debt. In our applications, τ will be large enough that this assumption is innocuous. All the quadratic variation terms are of the order $\mathcal{O}(dt)$ and henceforth we collapse them to an unspecified function $q(A_t, \sigma_{A,t}; f)$, where the notation captures the dependence of the higher order Itō terms on the partial derivatives of the call option pricing function.

In reality, we do not observe A_t as it is the market value of assets. However, given that the call option pricing function is monotonically increasing in its first argument, it is safe to assume that $f(\cdot)$ is invertible with respect to this argument. We further assume that the call pricing function is homogenous of degree one in its first two arguments, which is a standard assumption in the option pricing literature. Let us then

[6] Implicit is that the volatility process satisfies the usual restrictions necessary to apply Itō's Lemma.
[7] A short and certainly incomplete list includes Black and Scholes (1973), Heston (1993), and Bates (1996).

define the inverse call option formula as follows:

$$
\begin{aligned}
\frac{A_t}{D_t} &= g\left(E_t/D_t, 1, \sigma_{A,t}, \tau, r_t\right) \\
&\equiv f^{-1}\left(E_t/D_t, 1, \sigma_{A,t}, \tau, r_t\right)
\end{aligned} \tag{4}
$$

Thus, Equation (3) reduces returns to the following:[8]

$$
\begin{aligned}
\frac{dE_t}{E_t} &= \overbrace{\Delta_t \cdot g\left(E_t/D_t, 1, \sigma^f_{A,t}, \tau, r_t\right) \cdot \frac{D_t}{E_t}}^{\equiv LM(E_t/D_t, 1, \sigma_{A,t}, \tau, r_t)} \times \frac{dA_t}{A_t} + \frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + q(A_t, \sigma^f_{A,t}; f)dt \\
&= LM\left(E_t/D_t, 1, \sigma_{A,t}, \tau, r_t\right) \times \frac{dA_t}{A_t} + \frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + q(A_t, \sigma^f_{A,t}; f)dt
\end{aligned} \tag{5}
$$

For reasons that will become clear shortly, we call $LM\left(E_t/D_t, 1, \sigma^f_{A,t}, \tau, r_t\right)$ the "leverage multiplier". Henceforth, when it is obvious, we will drop the functional dependence of the leverage multiplier on leverage, etc. and instead denote it simply by LM_t. In order to obtain a complete law of motion for equity, we need to know the dynamics of volatility, $\sigma_{A,t}$, as opposed to variance. Itō's Lemma implies that the volatility process behaves as follows:

$$
\begin{aligned}
d\sigma_{A,t} &= \overbrace{\left[\frac{\mu_v(t, v_t)}{2\sigma_{A,t}} - \frac{\sigma_v^2(t, v_t)}{8\sigma^3_{A,t}}\right]}^{\equiv s(\sigma_{A,t}; \mu_v, \sigma_v)} dt + \frac{\sigma_v(t, v_t)}{2\sigma_{A,t}} dB_v(t) \\
&= s\left(\sigma_{A,t}; \mu_v, \sigma_v\right) dt + \frac{\sigma_v(t, v_t)}{2\sigma_{A,t}} dB_v(t)
\end{aligned} \tag{6}
$$

Plugging Equations (2), (6) into Equation (5) yields the desired full equation of motion for equity returns:

$$
\begin{aligned}
\frac{dE_t}{E_t} &= \left[LM_t \mu_A(t) + s\left(\sigma_{A,t}; \mu_v, \sigma_v\right) + q(A_t, \sigma_{A,t}; f)\right] dt \\
&\quad + LM_t \sigma_{A,t} dB_A(t) + \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} dB_v(t)
\end{aligned} \tag{7}
$$

Since our empirical focus will be on daily equity and asset returns, we ignore the drift term for equity.

[8] Using the fact that $f(\cdot)$ is homogenous of degree 1 in its first argument also implies that

$$\Delta_t = \partial f\left(A_t, D_t, \sigma_{A,t}, \tau, r\right)/\partial A_t = \partial f\left(A_t/D_t, 1, \sigma_{A,t}, \tau, r\right)/\partial(A_t/D_t)$$

So with an inverse option pricing formula, $g(\cdot)$ in hand we can define the delta in terms of leverage E_t/D_t.

Typical daily equity returns are virtually zero on average, so for our purposes ignoring the equity drift is harmless.[9] Instantaneous equity returns then naturally derive from Equation (7) with no drift:

$$\frac{dE_t}{E_t} = LM_t \sigma_{A,t} dB_A(t) + \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} dB_v(t) \tag{8}$$

Suppose for a moment that we can ignore the contribution of asset volatility shocks to equity returns.

Assumption 1. *For the purposes of daily equity return dynamics, we can ignore the following term in Equation (8):*

$$\frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} dB_v(t)$$

In Appendix A, we show that Assumption 1 is appropriate in a variety of option pricing models.[10] The intuition behind this result is as follows: mean reversion is embedded in any reasonable model of volatility. In this case, the time it takes volatility to mean revert is much shorter than typical debt maturities for firms. Thus, the cumulative asset volatility over the life of the option (equity) is effectively constant. In turn, the ν_t term is nearly zero, and so shocks to asset volatility get washed out as far as equity returns are concerned. In the Black-Scholes-Merton (henceforth BSM) case, this assumption holds exactly since asset volatility is constant. Under Assumption 1, equity returns and instantaneous equity volatility are given by:

$$\begin{aligned} \frac{dE_t}{E_t} &= LM_t \sigma_{A,t} dB_A(t) \\ vol_t\left(\frac{dE_t}{E_t}\right) &= LM_t \times \sigma_{A,t} \end{aligned} \tag{9}$$

Equation (9) is our key relationship of interest. It states that equity volatility (returns) is a scaled function of asset volatility (returns), where the function depends on financial leverage, D_t/E_t, as well as asset volatility over the life of the option (and the interest rate). The moniker of the "leverage multiplier" should be clear now: LM_t describes how equity volatility is amplified by financial leverage. It is illustrative to first explore the shape of the leverage multiplier; a natural benchmark to do so is within the Black-Scholes-Merton model.

[9] Indeed, ignoring the drift when thinking about long horizon asset returns (and levels) is not trivial.

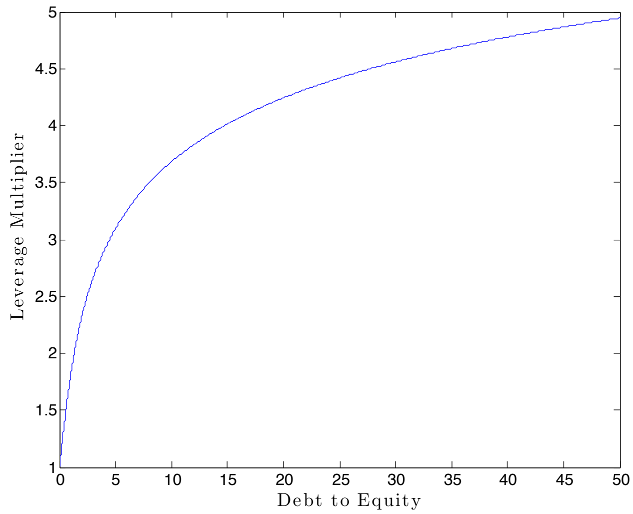
[10] e.g. When the underlying asset process has jumps, stochastic volatility, stochastic volatility and jumps, etc.

2.2 The Shape of the Leverage Multiplier

2.2.1 Leverage Multiplier in the Black-Scholes-Merton World

It is straightforward to compute $LM(\cdot)$ when the relevant option pricing model is BSM. To start, we fix annualized asset volatility to $\sigma_A = 0.15$, time to maturity of the debt $\tau = 5$, and the risk-free rate $r = 0.03$. Figure 1 plots the leverage multiplier against financial leverage (D_t/E_t) in this case:

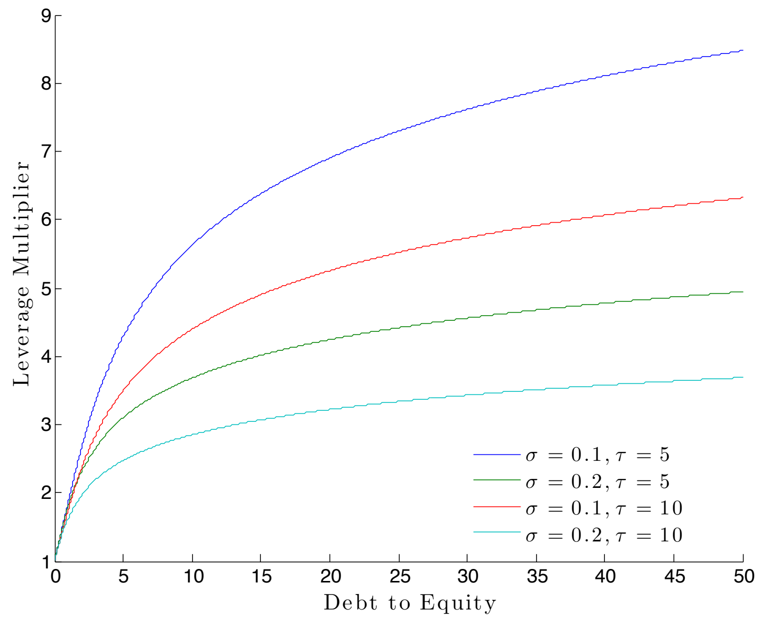
Figure 1: BSM Leverage Multiplier



Notes: This figure plots the leverage multiplier in the BSM model. Annualized asset volatility is set to $\sigma_A = 0.15$, the time to maturity of the debt is $\tau = 5$, and the annualized risk-free rate is $r = 0.03$

From Figure 1, we can see that the leverage multiplier is increasing in leverage. Intuitively, when a firm is more leveraged, its equity option value is further from the money, and asset returns amplify equity returns by a larger degree. Moreover, when leverage is zero (i.e. $D_t/E_t = 0$), the leverage multiplier is one, since it must be the case that assets are equal to equity. Next, we investigate how the BSM leverage multiplier changes as we vary the time to expiration and volatility:

Figure 2: BSM Leverage Multiplier with Varying σ_A and τ



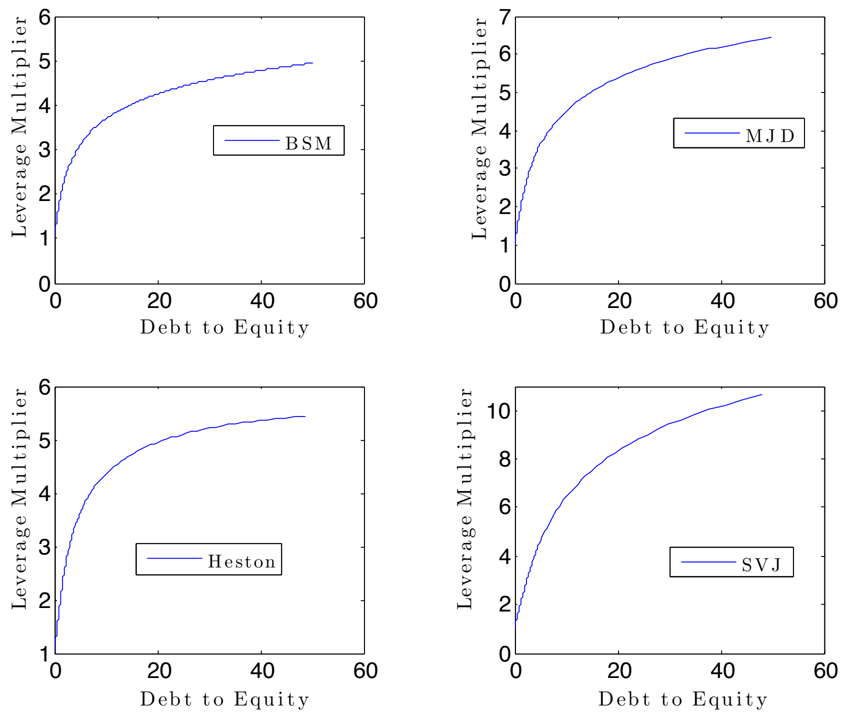
Notes: This figure plots the leverage multiplier in the BSM model. Annualized asset volatility takes on one of two values $\sigma_A \in \{0.1, 0.2\}$. The time to maturity of the debt also takes on two possible values $\tau \in \{5, 10\}$. The annualized risk-free rate is $r = 0.03$

Let us begin with the case where debt maturity is held constant, and volatility varies. When volatility increases, the leverage multiplier decreases. In this case, the likelihood that the equity is "in the money" rises with volatility and thus the effect of leverage on equity volatility is dampened. A similar argument holds when holding volatility fixed and varying debt maturity. Extending the maturity of the debt serves to dampen the leverage multiplier since, again, the equity has a better chance of expiring with value. The BSM model provides a useful benchmark in understanding the economics of the leverage multiplier, but it also provides a simple and easy way to compute a set of functions when evaluating $LM(\cdot)$. Our primary objective is to estimate a simple functional form for $LM(\cdot)$ that is not restricted to the assumptions of the BSM; however, we will ultimately be able to use the functions provided by BSM as a starting point for constructing a flexible specification for $LM(\cdot)$.

2.2.2 The Leverage Multiplier in Other Option Pricing Settings

The purpose of this subsection is to get a sense of the shape of the leverage multiplier in more complicated option pricing settings. Figure 3 provides a visual summary:

Figure 3: The Leverage Multiplier in Other Option Pricing Models



Notes: This figure plots the leverage multiplier in a variety of option pricing models. Full details of the construction can be found in Appendix A. The upper left panel is the benchmark BSM Model. The upper right panel is the Merton (1976) jump-diffusion model. The lower left panel is the Heston (1993) stochastic volatility model. Finally, the lower right panel is a stochastic volatility with jumps model that is used by Bates (1996) and Bakshi et al. (1997).

The full details for how we construct the leverage multiplier in each of the specific option pricing models can be found in Appendix A. In addition to the benchmark BSM case, Figure 3 plots the leverage multiplier in the Merton (1976) jump-diffusion model, the Heston (1993) stochastic volatility model, and the stochastic volatility with jumps model employed by Bates (1996) and Bakshi et al. (1997). The main takeaway is that, for a wide range of leverage, the shape of the leverage multiplier is roughly the same across option pricing models. So far, our exploration of the leverage multiplier has been in the context of continuous time; however, our eventual econometric model will fall under the discrete time GARCH class of models for assets.

In order to understand how the leverage multiplier behaves in this setting, we now turn to a Monte Carlo exercise involving GARCH option pricing.

2.2.3 The Appropriate Leverage Multiplier with GARCH and Non-Normality

Our Monte Carlo approach is motivated by option models estimated when the underlying follows a GARCH type process, as in Barone-Adessi, Engle, and Mancini (2008). When pricing options on GARCH processes, there is often no closed form solution for call prices, thus necessitating the use of simulation techniques. First, we assume a risk-neutral return process for assets. In our simulations, we adopt four different asset processes: 1) a GARCH process with normally distributed innovations, 2) a GARCH process with t distributed innovations, 3) an asymmetric GARCH process with normally distributed innovations, and 4) an asymmetric GARCH process with t distributed errors.[11] For completeness, we present these recursive volatility models below:

$$\begin{aligned} GARCH: \quad \sigma^2_{A,t} &= \omega + \alpha r^2_{A,t-1} + \beta \sigma^2_{A,t-1} \\ GJR: \quad \sigma^2_{A,t} &= \omega + \alpha r^2_{A,t-1} + \gamma r^2_{A,t-1} 1_{r_{A,t-1}<0} + \beta \sigma^2_{A,t-1} \end{aligned}$$

The GJR process captures the familiar pattern in equity returns of negative correlation between volatility and returns; this correlation is captured by the asymmetry parameter, γ. In our parameterization of these processes, we set the asymmetry parameter to be quite large, as this is one way to capture how risk-aversion affects the risk-neutral asset process. In addition, for the models with t-distributed innovations, we set the degrees of freedom to six in order to fatten the tails of the asset return process. In order to ensure comparability across models within our simulation, we change ω so that the unconditional volatility of all the processes is 15% annually. Table 1 summarizes our parametrization:

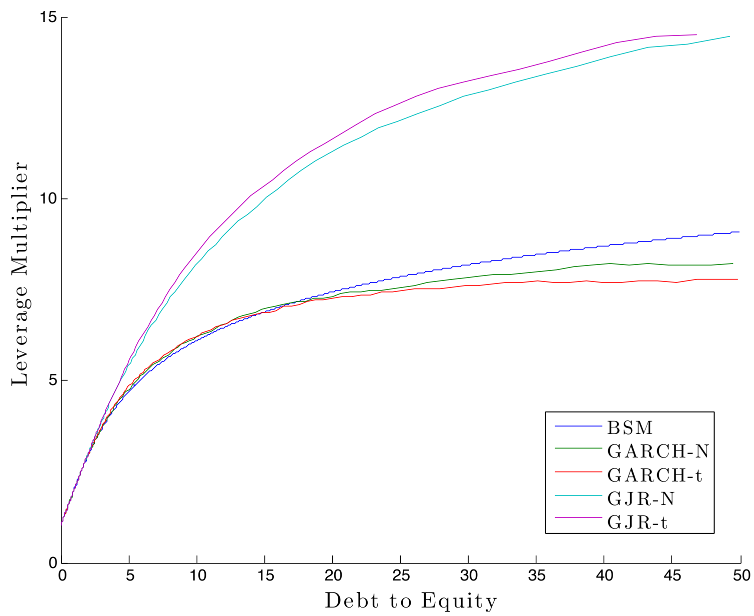
Model	**Parameter** α	γ	β
GARCH with Normal Errors	0.07	-	0.92
GARCH with t Errors	0.07	-	0.92
GJR with Normal Errors	0.022	0.18	0.884
GJR with t Errors	0.022	0.18	0.884

Table 1: Parameterizations for Simulated Asset Processes

[11] The asymmetric GARCH process we use is the so-called GJR process of Golston et al. (1992).

For each process, we simulate the asset process 10,000 times from an initial asset value of $A_0 = 1$. We assume the debt matures in two years and, for simplicity, set the risk-free rate to zero. The simulation generates a set of terminal values, A_T, which in turn generate an equity value for each value of debt D.[12] We then compute numerical derivatives to measure how the equity value changes with respect to A_0. Finally, we calculate the leverage multiplier implied by each asset return process and plot it against the implied financial leverage in Figure 4:

Figure 4: Simulated Leverage Multiplier in Stochastic Volatility and Non/Normality



Notes: The figure plots the simulated leverage multiplier under different asset return process specifications. We consider GARCH and GJR process, each with normally distributed and t distributed errors. The unconditional volatility in all the models is 15% annually, the time to maturity of the debt is two years, and the risk-free rate is set to zero.

The economics behind the shape of the leverage multiplier under various asset return processes are subtle. The benchmark case of BSM is given by the blue line in Figure 4, and it is easy to see that in a symmetric setting, making the tails of the asset distribution longer via GARCH decreases the leverage multiplier for larger values of debt (the green and red lines). For larger values of debt, extending the tails of the asset distribution serves the same function as increasing volatility in the BSM case. When we introduce asset

[12] i.e. $E = \frac{1}{10{,}000} \sum_{i=1}^{10{,}000} \max(A_{T,i} - D, 0)$, where i is the index for each simulation run. Varying D thus generates a variable range of leverage, D/E.

volatility asymmetry via the GJR process, the leverage multiplier increases dramatically relative to the BSM benchmark (turquoise and purple lines). Volatility asymmetry effectively makes the figure asset distribution left skewed, which shortens the right tail of the distribution and increases the leverage multiplier. In this case, leverage has a larger amplification on equity volatility because high leverage corresponds to a much smaller likelihood the equity expires "in the money". In general, it is clear that the shape of the leverage multiplier is robust across a variety of continuous time and discrete time option pricing models. With this in mind, we propose a parameterized function to capture leverage amplification mechanisms in a relatively "model-free" way.

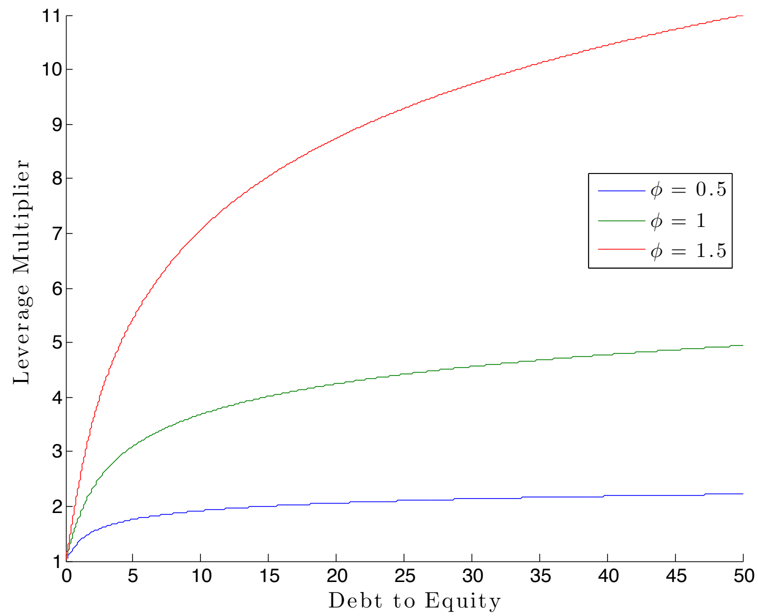
2.3 A Flexible Leverage Multiplier

In the derivation of Equation (9), we did not assign specific functions to $g(\cdot)$ and Δ_t. Define $g^{BSM}(\cdot)$ and Δ_t^{BSM} as the BSM inverse call and delta functions, respectively. We then propose the following specification for the leverage multiplier:

$$LM\left(D_t/E_t, \sigma^f_{A,t}, \tau, r_t; \phi\right) = \left[\Delta_t^{BSM}\left(E_t/D_t, 1, \sigma^f_{A,t}, \tau, r_t\right) \times g^{BSM}\left(E_t/D_t, 1, \sigma^f_{A,t}, \tau, r_t\right) \times \frac{D_t}{E_t}\right]^{\phi} \quad (10)$$

In this case, ϕ is the departure from the BSM model. When taking our model to data, it will be an estimated parameter. One advantage of our proposed leverage multiplier in (10) is its simplicity in terms of computation, as the BSM delta and inverse call functions are well-known. Raising the BSM leverage multiplier to an arbitrary power also preserves a necessary condition for $LM(\cdot)$ to have a value of one when leverage is zero. It is worth emphasizing that our leverage multiplier simply uses a mathematical transformation of the BSM functions. For example, in a BSM world, $g^{BS}(\cdot)$ would be interpreted as the asset to debt ratio, whereas for our model it is simply a function. Similarly, the BSM Δ_t^{BSM} in our specification is not interpreted as the correct hedge ratio, but again merely serves as a function for our purposes. Let us now examine how our leverage multiplier changes for different values of ϕ, which we plot below in Figure 5:

Figure 5: Leverage Multiplier for Different Values ϕ



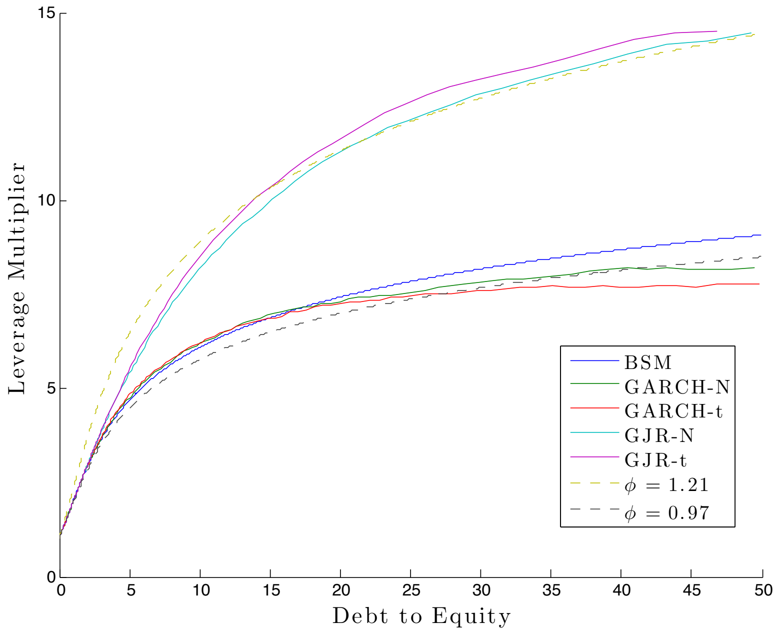
Notes: This figure plots the leverage multiplier according to the specification in (10) for different values of ϕ. In our baseline case, the annualized σ_A is held constant at 0.15, $\tau = 5$, and $r = 0.03$.

Unsurprisingly, increasing ϕ increases the leverage multiplier. In addition, varying ϕ preserves the concavity and montonicity of the BSM leverage multiplier.[13] For firms with a low value of ϕ, high levels of leverage have a small amplification effect in terms of equity volatility. Building on the intuition from the BSM case, for these firms leverage plays a small role in the moneyness of the equity, which likely corresponds to healthier firms. The converse holds true as well, as firms with high ϕ experience large equity volatility amplification, even for low levels of financial leverage.

To highlight the flexibility of our specification, we revisit the Monte Carlo exercise from Section 2.2.3. That is, Figure 6 plots the leverage multiplier in a GARCH option pricing setting, as well as our leverage multiplier for a few different values of ϕ:

[13]To be precise, ϕ preserves the concavity so long as it is not too large, long-run asset volatility is not too small, and τ is not too small. In practice, this is not an issue, even for financial firms who have larger amounts of leverage. We explore these issues further in the Online Appendix.

Figure 6: Simulated Leverage Multiplier and Our Specification



Notes: The figure plots the simulated leverage multiplier under different asset return process specifications. We consider GARCH and GJR process, each with normally distributed and t distributed errors. The unconditional volatility in all the models is 15% annually, the time to maturity of the debt is two years, and the risk-free rate is set to zero. In addition, we plot our leverage multiplier from specification (10) for different values of ϕ to demonstrate that our model captures various asset return processes well.

The straightforward takeaway from Figure 6 is that varying ϕ in our leverage multiplier specification captures various asset return processes very well. Increasing ϕ is successful in matching the patterns in the leverage multiplier that arise in stochastic volatility, asymmetric volatility, and non-normal settings. Our specification then naturally gives rise to the full Structural GARCH model.

2.4 The Full Recursive Model

The analysis from the Section 2 motivates the use of our leverage multiplier in describing the relationship between equity volatility and leverage. To make the model fully operational in discrete time, we propose the following process for equity returns:

$$\begin{aligned}
r_{E,t} &= LM_{t-1} r_{A,t} \\
r_{A,t} &= \sqrt{h_{A,t}} \varepsilon_{A,t}, \quad \varepsilon_{A,t} \sim D(0,1) \\
h_{A,t} &= \omega + \alpha \left(\frac{r_{E,t-1}}{LM_{t-2}} \right)^2 + \gamma \left(\frac{r_{E,t-1}}{LM_{t-2}} \right) 1_{r_{E,t-1}<0} + \beta h_{A,t-1} \\
LM_{t-1} &= \left[\triangle_{t-1}^{BSM} \times g^{BSM} \left(E_{t-1}/D_{t-1}, 1, \sigma_{A,t-1}^{f}, \tau \right) \times \frac{D_{t-1}}{E_{t-1}} \right]^{\phi}
\end{aligned} \tag{11}$$

Henceforth, we will call the specification described in Equation (11) as a "Structural GARCH" model. The parameter set for the Structural GARCH is $\Theta = (\omega, \alpha, \gamma, \beta, \phi)$, so there is only one extra parameter compared to a vanilla GJR model. We will confront the issue of how to compute τ and $\sigma_{A,t-1}^{f}$ in the next section when describing the data and estimation techniques used in our empirical work. We also introduce lags in the appropriate variables (e.g. the leverage multiplier) to ensure that one-step ahead volatility forecasts are indeed in the previous day's information set. The model in (11) nests both a simple GJR model ($\phi = 0$) and the BSM model ($\phi = 1$), and thus provides a statistical test of how leverage affects equity volatility.[14] Importantly, the equity return series will inherit volatility asymmetry from the asset return series; this is an important feature of equity returns in the data.[15] The recursion for equity returns (and asset returns) in (11) is simple and straightforward to compute, yet powerful. For example, when simulating this model, if a series of negative asset returns is realized (and hence negative asset returns since they share the same shock), volatility rises due to the asymmetric specification inherent in the GJR. In that case, leverage also rises, thereby increasing the leverage multiplier and resulting in an even stronger amplification effect for equity volatility. As we saw in the recent financial crisis, this was a key feature of the data, particularly for highly leverage financial firms. Additionally, by letting ϕ vary from firm to firm, we effectively allow a different option pricing model to apply to the capital structure of each firm. This flexibility is difficult to achieve if we impose an option pricing model on the data a priori since, as we showed, ϕ allows us to move across different *classes* of option pricing models. Furthermore, to the extent that our leverage multiplier form captures various option pricing models, the Structural GARCH allows us to infer a high frequency asset

[14] $\phi = 1$ nests the BSM exactly if we use a constant forecast of asset volatility over the lifetime of the option. As mentioned, we estimate this model against a model where we use a GJR forecast for $\sigma_{A,t}^{f}$. The results are very similar so we refer between the two without distinction.

[15] For example, it is has been shown that a GJR process for equity can replicate features of equity option data like the volatility smirk.

return series with stochastic volatility in a relatively model-free way. Later, this will prove to be extremely useful for a number of applications of the model.

3 Data Description and Estimation Details

We now turn to estimating the Structural Model using equity return data. In order to compute the leverage multiplier, we also need balance sheet information, which we obtain from Bloomberg. In particular, we define D_t as the book value of debt at time t. In order to avoid estimation issues inherent with quarterly data, we smooth the book value of debt using an exponential average with smoothing parameter of 0.01. This smoothing parameter value implies a half-life of approximately seventy days in terms of the weights of the exponential average; this is reasonable for quarterly data. When estimating the full model, we use quasi-maximum likelihood and the associated standard errors for parameter estimates. In order to ensure a global optimum is reached, we also conduct each maximum likelihood optimization over a grid of twenty-four different starting values.[16]

The set of firms we analyze are financial firms.[17] The reason we focus on financial firms is twofold: first, these firms typically have extraordinarily high leverage and structural models have failed to model these firms well. Second, given the high volatility in the recent crisis that was accompanied by unprecedented leverage, this set of firms presents an important sector to model from a systemic risk and policy perspective. To this end, one of the applications of our model that we will explore in later sections involves systemic risk measurement of financials. In future work, we hope to extend the set of firms we analyze. The remaining issues are how to treat both the time to maturity of the debt τ and the asset volatility over the life of the debt $\sigma^f_{A,t}$.

Time to Maturity of the Debt The leverage multiplier requires as an input a time to maturity of the debt. As the book value of debt combines a number of different debt maturities, we simply iterate over different τ during estimation. Specifically, we estimate the model for $\tau \in [1, 30]$, restricting τ to take on integer values. We keep the version of the model that attains the highest log-likelihood function.[18]

[16]The Matlab code for estimation of the model via QMLE with the correct standard errors is available upon request.
[17]A full description of the set of firms is contained in the Appendix.
[18]TO DO: Confidence intervals for τ using concentrated likelihood function.

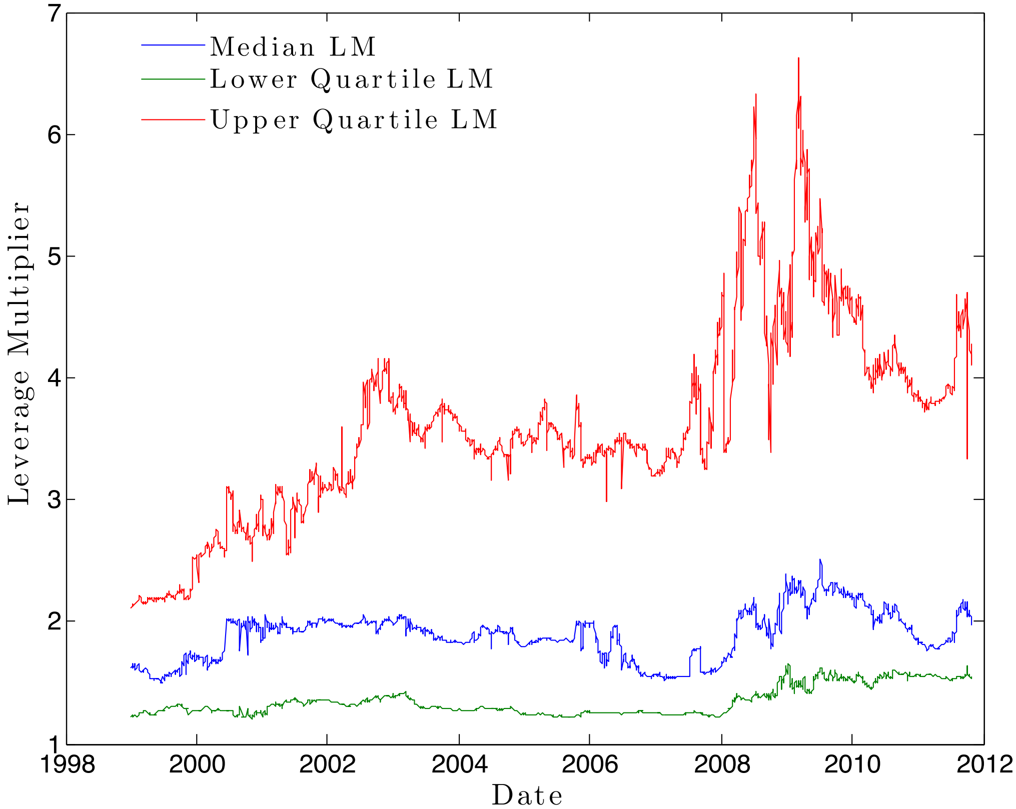
Asset Volatility over Life of Debt We take two different approaches for the computing the value of $\sigma^f_{A,t}$. The first is to use the unconditional volatility implied by the asset volatility series corresponding to the unconditional volatility of a GJR process. Using a constant $\sigma^f_{A,t}$ in fact completely eliminates any issues in ignoring the "vega" terms in our motivating derivation of the leverage multiplier (see Equation (3)). The second approach is to use the GJR forecast over the life of the debt at each date t. It is straightforward to derive the closed form expression for this forecast. In estimating the full model, we use both approaches for $\sigma^f_{A,t}$ and choose the model with the highest likelihood. That is, we estimate the model over all ranges of τ for each type of $\sigma^f_{A,t}$ and choose the model with the highest likelihood.

4 Empirical Results

4.1 Cross-Sectional Summary

We begin by presenting a cross-sectional summary of the estimation results. Since the main contribution of this paper is the leverage multiplier, Figure 7 plots the estimated time-series of the lower quartile, median, and upper quartile leverage multipliers, across all firms.

Figure 7: Time Series of Leverage Multiplier Across Quartiles



As we can see, there is considerable cross-sectional heterogeneity in the leverage multiplier, even within financial firms. It appears that across all firms, the leverage multiplier moves with the business cycle, which is not surprising given that leverage itself tends to do so as well. In the top quartile of firms leverage amplifies equity volatility by a factor of eight during the financial crisis. Evidently, for this set of firms, the leverage amplification mechanism has remained high in the years following the crisis.

Table 1 provides cross-sectional summary statistics for the point estimates of the Structural GARCH model.

Parameter	**Median**	**Median t-stat**	**% with** $|t| > 1.64$
ω	1.0e-06	1.43	30.9
α	0.0442	3.16	85.2
γ	0.0674	2.50	72.8
β	0.9094	71.21	98.8
ϕ	0.9876	2.87	75.3

Table 2: Cross-Sectional Summary of Structural GARCH Parameter Estimates

In our model, the first four estimates represent the GJR parameters for the asset return series. It is not surprising then that they resemble those found in equity returns. The parameter ω is an order of magnitude smaller than usual, but this is natural given asset returns are less volatile than equity returns and ω is a determinant of the unconditional volatility. The asset process is indeed stationary as seen by the combination of α, γ, β and standard results on the stationarity of GARCH processes. One subtle, but key, difference in the current estimates is the parameter γ, which is higher than it is for equity returns in this subset of stocks. Recall that γ dictates the correlation between volatility and returns, and thus it appears that the volatility asymmetry we observe in equity is somewhat dampened in asset returns. In one application of the model, we will explore this idea further as it pertains to the classical leverage effect of Black (1976) and Christie (1982).

The new parameter in our model is ϕ. As is evident from the third and fourth columns of Table 2, ϕ is statistically different than zero for a majority of firms. Therefore, the effect of leverage on equity volatility via our leverage multiplier appears to be substantial for a large number of financial firms. Interestingly, the median ϕ is close to one, as the BSM model would suggest. These results are consistent with the findings of Schaefer and Strebulaev (2008) who find that while the Merton (1974) model does poorly in predicting the levels of credit spreads, it is successful in generating the correct hedge ratios across the capital structure of the firm. In our context, we interpret their finding and our estimation of ϕ to mean that we are able to recover the daily returns of assets well, even if we cannot pinpoint the level of assets.

4.2 Aggregation

4.2.1 Aggregate Leverage Multiplier

We aggregate our results across firm by creating three indices: 1) a value-weighted average equity volatility index, 2) a value-weighted average asset volatility index, and 3) an aggregate leverage multiplier. The aggregate leverage multiplier is simply the ratio of the equity volatility index to the asset volatility index. The weights used in creating each respective index are derived from equity valuations. Figure 8 plots these three time series:

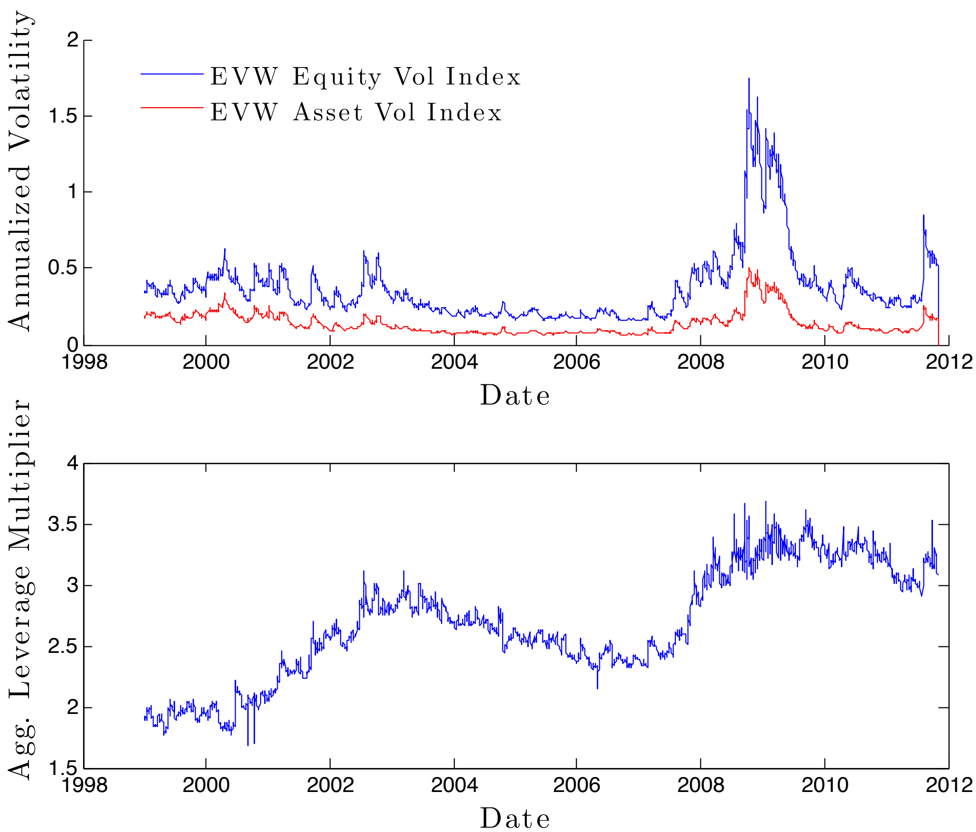


Figure 8: Aggregate Equity Volatility, Asset Volatility, and Leverage Multiplier

Again, it is clear that there is a cyclicality in the aggregated leverage multiplier. A pressing issue in the wake of the financial crisis is the role of leverage and the health of the financial sector. Since our model provides estimates of leverage amplification in terms of equity volatility (as well as asset volatility), we focus on these aggregated time-series through the financial crisis:

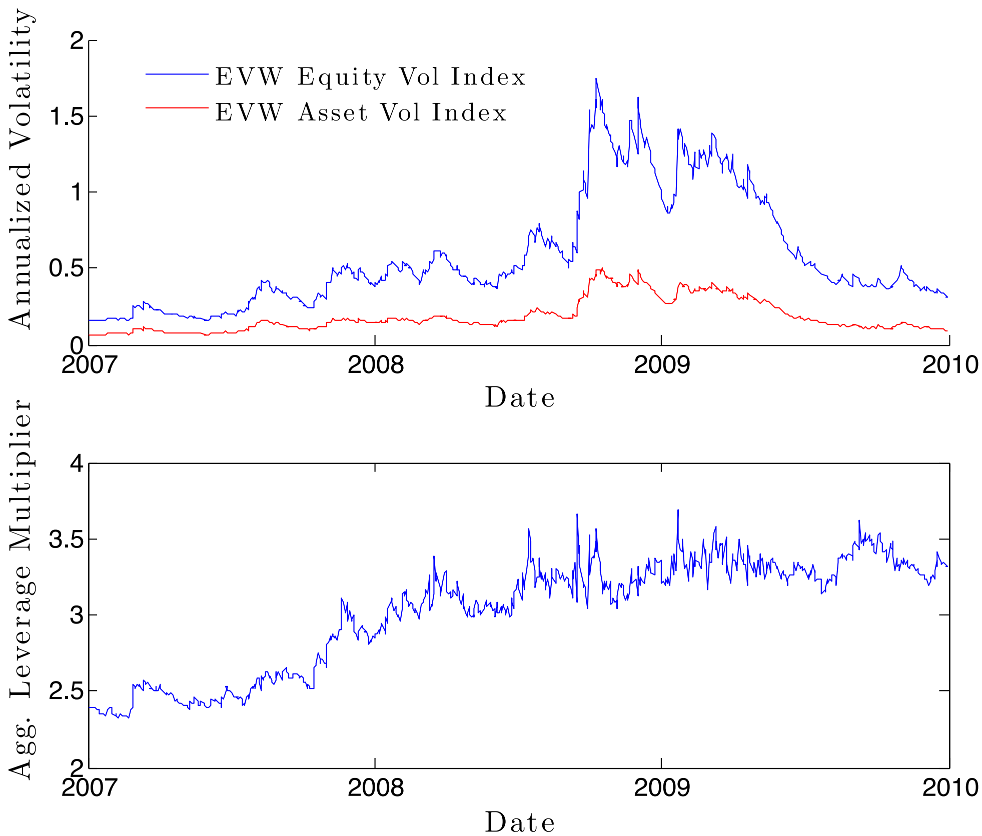


Figure 9: Aggregate Equity Volatility, Asset Volatility, and Leverage Multiplier During Financial Crisis

It is clear that the rise in equity volatility for the aggregate financial sector began in the summer of 2007. However the rise in asset volatility did not really occur until late in 2008. The increase in leverage in 2007 was partly an increase in aggregate liabilities and partly a fall in equity valuation. After the fall of Lehman, asset volatility rose dramatically as well and the leverage multiplier continued to rise stabilizing in the spring of 2009.

5 Applications

In this section we explore two applications of the Structural GARCH: 1) the leverage effect and 2) systemic risk measurement.

5.1 The Leverage Effect

The leverage effect of Black (1976) and Christie (1982) documents the negative correlation that exists between equity returns and equity volatility. One possible explanation for this stylized fact is that when a firm experiences a fall in equity, its financial leverage mechanically rises, the company becomes riskier, and hence volatility rises. A second explanation points to the role of risk premia in describing the negative correlation between equity returns and equity volatility (e.g. French et al. (1987)). In this explanation, a rise in future volatility raises the required return on equity, leading to an immediate decline in the stock price. The Structural GARCH model provides a natural framework to explore these issues econometrically.[19]

Recall that the Structural GARCH model delivers an estimate of the daily return of assets.[20] Any correlation between asset volatility and asset returns can not be due to financial leverage; so, if a correlation does exist, it must be attributed to a risk-premium argument. When applying the GJR volatility model to a given time series of returns, the γ parameter is one way to measure the correlation between the volatility and returns (e.g. a higher γ corresponds to more negative correlation). Therefore, we would expect the GJR γ estimated from equity returns to be larger than the same parameter estimated from asset returns. Indeed, the median γ for equity returns is 0.0846 and the median γ for asset returns is 0.0674; thus, for our subsample of firms, financial leverage accounts for roughly 23% of the so-called leverage effect.

To put a bit more structure on the implications of Structural GARCH and the leverage effect, we run the following cross-sectional regression:

$$\gamma_{E,i} - \gamma_{A,i} = a + b \times \overline{D/E}_i + error_i \tag{12}$$

where $\gamma_{E,i}$ and $\gamma_{A,i}$ are the estimated GJR asymmetry parameter for firm i's equity returns and firm i's asset returns, respectively. $\overline{D/E}_i$ is the median debt to equity ratio for firm i over the sample period. The logic behind the regression in (12) is simple: firms with higher leverage should experience a larger reduction in volatility asymmetry after unlevering the firm. Table 3 presents the results:

[19] Other econometric studies of the leverage effect include Bekaert and Wu (2001). A key divergence in our approach is that we allow debt for the firm to be risky, as in the Merton (1974) model.

[20] Again, this relies on a few assumptions. First, our specification ignores the effect of changes in long-run asset volatility on daily equity returns. Second, we assume that the book value of debt adequately captures the outstanding liabilities of the firm. For example, we do not consider non-debt liabilities in our baseline specification. Still, the Structural GARCH model is, at worst, effective in at least partially unlevering the firm.

Variable	Coefficient Value	t-stat	R^2
b	0.0029	4.0471	17.8%

Table 3: Equity Asymmetry versus Asset Asymmetry

Notes: This table presents the cross-sectional regression described in Equation (12).

As expected, firms with higher median leverage have a larger gap between their equity and asset asymmetry. As we saw before, there is still a substantial amount of asset volatility asymmetry (i.e. the γ parameter for assets is 0.0674). At the asset level, firms with higher volatility asymmetry should have higher risk premia. Thus, as a rough quantitative exercise, we run the following two-stage regression:

$$\begin{aligned} \text{Stage 1:} \quad r^A_{i,t} &= c + \beta^A_{mkt,i} r^E_{mkt,t} + e_{i,t} \\ \text{Stage 2:} \quad \gamma_{A,i} &= e + f \times \beta^A_{mkt,i} + \varepsilon_i \end{aligned} \tag{13}$$

where r^E_{mkt} is the return on the equity market index. Stage 1 of the regression is designed to deliver a measure of firms's risk premia, i.e. its CAPM beta.[21] The coefficient f in the Stage 2 regression is the main variable of interest. A positive value corroborates the risk premium story for volatility asymmetry. The results of the two-stage regression are found in Table 4:

Variable	Coefficient Value	t-stat	R^2
f	0.0287	1.98	4.95%

Table 4: Risk-Premium Effect on Asset Asymmetry

Notes: This table presents the two-stage regression results in Equation (13). The first stage regression estimates, for each firm's asset return series, the equity market beta. The second stage regresses a measure of asset volatility asymmetry, the GJR asset γ, on the regression coefficient from Stage 1.

Unsurprisingly, firms with higher market betas have higher asset volatility asymmetry. Though the results are not overwhelming, we view them as qualitative confirmation for how the Structural GARCH unlevers the firm. We now turn to using the Structural GARCH model to measure systemic risk.

[21] Note that since we are interested in the cross-sectional behavior of $\gamma_{A,i}$ we are not concerned with using the return on the equity market. If, for example, we used some proxy for the a broad market asset market index, only the magnitude of the coefficient f would change.

5.2 Systemic Risk Measurement

Given the unprecedented rise in leverage and equity volatility during the financial crisis of 2007-2009, systemic risk measurement is a natural application of the Structural GARCH model. To see why, consider the following thought experiment. Following a negative shock to equity value, the financial leverage of the firm mechanically rises. In a simple asymmetric GARCH model for equity, the rise in volatility following a negative equity return is invariant to the capital structure of the firm. However, in the Structural GARCH model, the leverage multiplier will be higher following a negative equity return; thus, equity volatility will be more sensitive to even slight rises in asset volatility. In simulating the model, this mechanism would manifest itself if the firm experiences experiences a sequence of negative asset shocks. Due to asset volatility asymmetry, a sequence of negative asset returns increases asset volatility. In turn, there will potentially be explosive equity volatility since the leverage multiplier will be large in this case. Casual observation of equity volatility and leverage during the crisis clearly supports such a sequence of events.

In order to embed this appealing feature of the Structural GARCH model into systemic risk measurement, we adapt the SRISK metric of Brownlees and Engle (2012) and Acharya et al. (2012). The reader should refer to these studies for an in-depth discussion of SRISK; to keep the paper relatively self-contained, we will provide a brief summary. Qualitatively, SRISK is an estimate of the amount of capital that an institution would need in order to function normally in the event of another financial crisis. To compute SRISK, we first compute a firm's marginal expected shortfall (MES), which is the expected loss of a firm when the overall market declines a given amount over a given time horizon.[22] In turn, MES requires us to simulate a bivariate process for the firm's equity return, denoted $r^E_{i,t}$, and the market's equity return, denoted $r^E_{m,t}$. The bivariate process we adopt is described as follows:

$$\begin{aligned} r^E_{m,t} &= \sqrt{h^E_{m,t}}\varepsilon_{m,t} \\ r^E_{i,t} &= \sqrt{h^E_{i,t}}\left(\rho_{i,t}\varepsilon_{M,t} + \sqrt{1-\rho^2_{i,t}}\xi_{i,t}\right) \\ &= LM_{i,t-1}\sqrt{h^A_{i,t}}\left(\rho_{i,t}\varepsilon_{M,t} + \sqrt{1-\rho^2_{i,t}}\xi_{i,t}\right) \\ (\varepsilon_{m,t}, \xi_{i,t}) &\sim F \end{aligned} \tag{14}$$

[22] Acharya et al. (2012) provide an economic justification for why marginal expected shortfall is the proper measure of systemic risk in the banking system.

where the shocks $(\varepsilon_{m,t}, \xi_{i,t})$ are independent and identically distributed over time and have zero mean, unit variance, and zero covariance. We do not assume the two shocks are independent, however, and allow them to have extreme tail dependence nonparametrically.[23] The processes $h^E_{m,t}, h^E_{i,t}$ and $\rho_{i,t}$ represent the conditional variance of the market, the conditional variance of the firm, and the conditional correlation between the market and the firm, respectively. It is also worth emphasizing that, due to the Structural GARCH model, we are really estimating correlations between shocks to the equity market index and shocks to firm asset returns. Generically, once the bivariate process in (14) is fully specified, we compute a 6-month MES (henceforth LRMES for "long-run" marginal expected shortfall) by simulating the joint processes for the firm and the market (with bootstrapped shocks) and conditioning on the event that the market declines by 40%. Incorporating the Structural GARCH model into LRMES is therefore very simple. As stated in Equation (14), we simply assume the volatility process for firm equity returns follows a Structural GARCH model. Finally, we assume that equity market volatility follows a familiar GJR(1,1) process and that correlations follow a DCC(1,1) model.

Once we have an estimate for the LRMES of a firm on a given day, we compute its capital shortfall in a crisis as follows:

$$CS_{i,t} = kDebt_{i,t} - (1-k)(1 - LRMES_{i,t})E_{i,t} \tag{15}$$

where $Debt_{i,t}$ is the book value of debt outstanding on the firm, $E_{i,t}$ is the market value of equity, and k is a prudential level of equity relative to assets. In our applications, we take $k = 8\%$ and, as is conventional in risk metrics such as VaR, we use positive values of $LRMES_{i,t}$ to represent declines in the firm's value. For example, if firm i is expected to lose 60% of its equity in a crisis, its LRMES will be 60%. Thus, positive values of capital shortfall mean that the firm will be short of capital in a crisis. Finally, we define the SRISK of a firm as:

$$SRISK_{i,t} = \max(CS_{i,t}, 0)$$

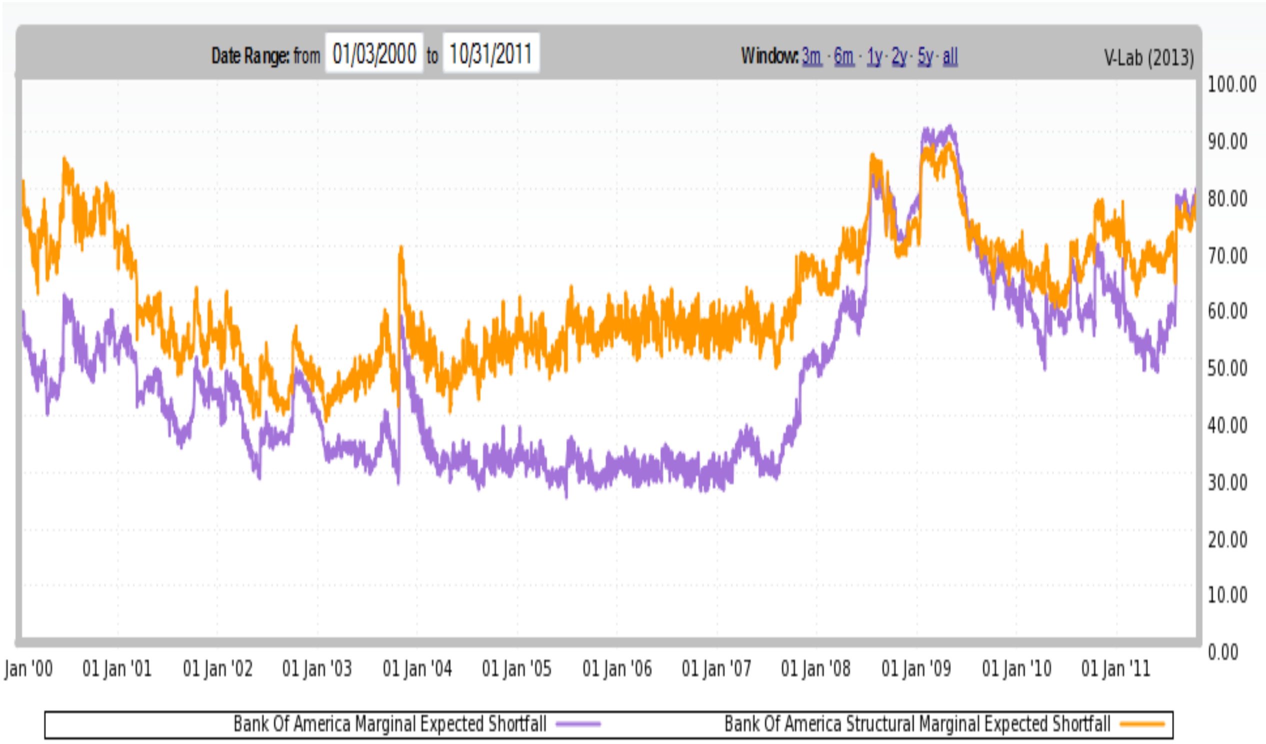
The parameters governing the market volatility and firm-market correlation are estimated recursively and allowed to change daily. However, due to the computational burden of estimating the Structural GARCH recursively each day, we use the full sample to estimate the Structural GARCH parameters. In future versions of SRISK measurement with Structural GARCH, we hope to estimate all parameters of the bivariate process

[23] See Brownlees and Engle (2012) for complete details.

recursively.[24] In the interest of brevity, we choose to focus on one firm: Bank of America.

To start, Figure 10 plots the $LRMES$ for Bank of America under using both the Structural GARCH, as well as a vanilla GJR model of univariate returns.

Figure 10: $LRMES$ for Bank of America



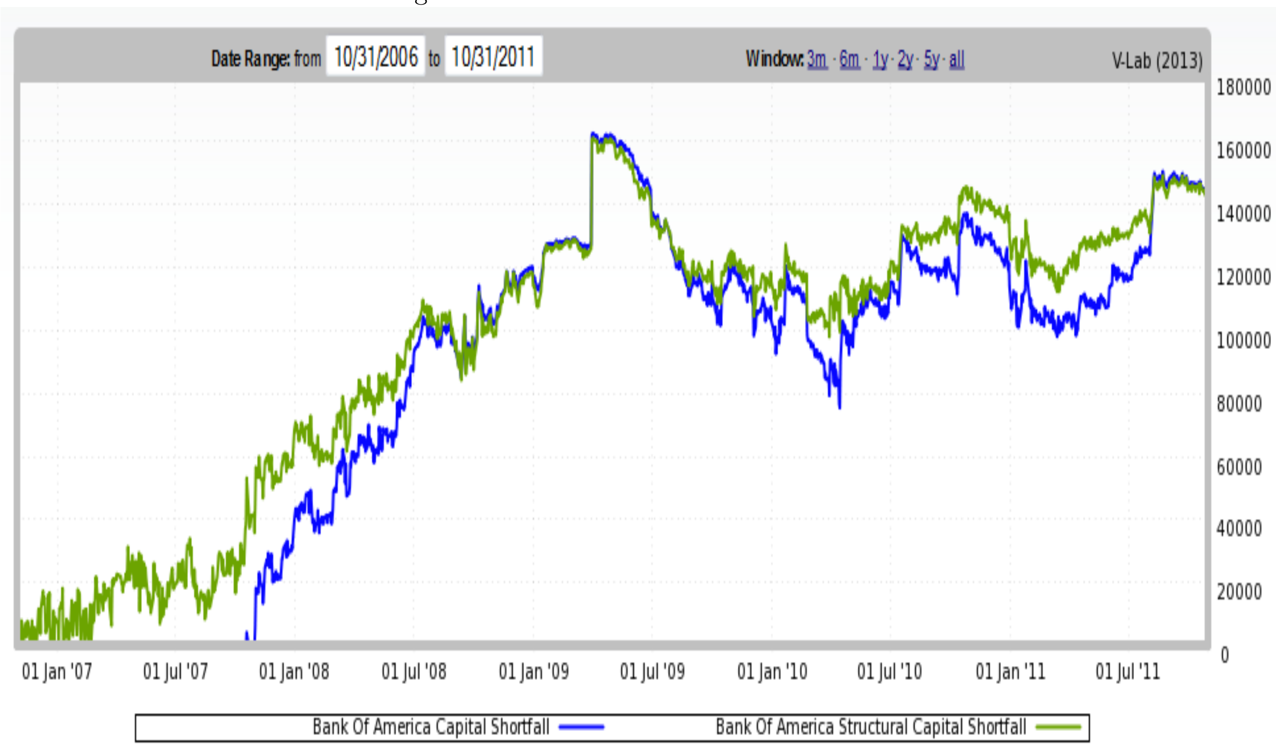
Notes: The figure above plots the Long Run Marginal Expected Shortfall of Bank of America. The purple line uses a standard GJR model for returns, whereas the orange line is the same calculation using the Structural GARCH model.

It is obvious that the Structural GARCH induces a higher MES than a standard asymmetric volatility model. The reasons for this pattern are due to the leverage amplification mechanism that is built into the model directly. In the low-volatility period from 2004-2007, the Structural GARCH model delivers a $LRMES$ that is nearly double the value that comes from a standard GJR. Even in this period of low leverage, there are negative equity paths in the Structural GARCH model that result in increases in leverage, which in turn result in higher volatility and therefore paths where equity suffers large losses. There is much less

[24] The calculation of SRISK for the full set of firms analyzed in Section 3 can be found at the following web address: INSERT LINK.

of a scope for this type of leverage spiral in low volatility/leverage periods in our typical volatility models. To focus in on the recent financial crisis, we translate our $LRMES$ calculations into $SRISK$ and plot the resulting series from starting in 2007 below:

Figure 11: $SRISK$ for Bank of America



Notes: The figure above plots the $SRISK$ of Bank of America from January 2007 to August 2011. The units of the y-axis are millions of USD. The blue line uses a standard GJR model for returns, whereas the green line is the same calculation using the Structural GARCH model.

Figure 11 illustrates why the Structural GARCH model may provide useful in terms of providing early warning signals of threats to financial stability. The results echo the dynamics of $LRMES$ under the Structural GARCH specification versus a standard GJR model. As early as January 2007, the SRISK (using Structural GARCH) of Bank of America starts to rise and hovers around twenty billion dollars. On the other hand, SRISK derived from a standard asymmetric volatility model shows no (expected) capital shortfalls until late 2007 and early 2008. Again, the success of the Structural GARCH along this dimension rests with the inherent leverage-volatility connection within the model. Qualitatively, the volatility and leverage link is apparent and has been discussed extensively in the media and the academic literature. Quantitatively, this link has been hard to pin down. The results in Figure 11 are evidence that the Structural GARCH model is, at very least, a partial resolution of this issue.

6 Conclusion

This paper has provided a econometric approach to disentangle the effects of leverage on equity volatility. The Structural GARCH model we propose is rooted in the classical Merton (1974) structural model of credit, but departs from it in a flexible and parsimonious way. In doing so, we are able to deliver high frequency asset return and asset volatility series. While we applied the model in two different settings (asymmetric volatility and systemic risk measurement), there are a variety of applications of the model in both corporate finance, asset pricing, and financial intermediation. For example, a natural asset pricing application is to use our estimates of daily asset returns and volatility to price other contingent claims, such as credit default swaps. In corporate finance, a interesting avenue of research would be to examine optimal leverage ratios, taking into account the effect that the leverage multiplier has on equity volatility. In other work, we are analyzing optimal government capital injections from a Structural GARCH lens. The Structural GARCH model is a simple econometric framework to explore all of these ideas.

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A Appendix: The Leverage Multiplier and Assumption 1 Under Different Option Pricing Models

In this appendix, we do three things. First, we add jumps to the asset process. Second, we explore whether stochatic volatility and/or jumps change the conclusions drawn in the main text regarding using Equation (9) as a model of equity returns. Finally, we confirm that jumps and stochastic volatility do not change the general shape of the leverage multiplier, and so our specification remains flexible enough to be accurate in these settings. Through this exposition, the steps to construct Figure 3 will also become clear. In order to accomplish these tasks, it will be useful to essentially start from scratch, and assign a generalized process for assets. As such, the reader may find many parts repetitive from the main text, but we take this approach for the sake of completeness.

A.1 Motivating the Model

Adding jumps means we re-define the equity value as follows:

$$E_t = f\left(A_t, D_t, \sigma_{A,t}, J_A, N_A, \tau, r_t\right) \tag{16}$$

where $f(\cdot)$ is an unspecified call option function, A_t is the current market value of assets, D_t is the current book value of outstanding debt, $\sigma_{A,t}$ is the (potentially stochastic) volatility of the assets. τ is the life of the debt, and finally, r_t is the annualized risk-free rate at time t. Additionally, J_A and N_A are processes that describe discontinuous jumps in the underlying assets. Next, we specify the following generic process for assets and variance:

$$\begin{aligned}
\frac{dA_t}{A_t} &= [\mu_A(t) - \lambda\mu_J]dt + \sigma_{A,t}dB_A(t) + J_A dN_A(t) \\
d\sigma^2_{A,t} &= \mu_v(t, \sigma_{A,t})dt + \sigma_v(t, \sigma_{A,t})dB_v(t)
\end{aligned} \tag{17}$$

where $dB_A(t)$ is a standard Brownian motion. $\sigma_{A,t}$ captures time-varying asset volatility. We also capture potential jumps in asset values via J_a and $N_A(t)$. $\log(1 + J_A) \sim N\left(\log[1 + \mu_J] - \sigma_J^2/2, \sigma_J^2\right)$ and N_t is a Poisson counting process with intensity λ. The relative price jump size J_A determines the percentage change in the asset price caused by jumps, and the average asset jump size is μ_J. We assume the jump size, J_A, is independent of $N_A(t), B_A(t)$, and $B_v(t)$. Similarly, the asset Poisson counting process $N_A(t)$ is assumed to be independent of $B_A(t)$ and $B_v(t)$. We allow an arbitrary instantaneous correlation of ρ_t between the shock to asset returns, $dB_A(t)$, and the shock to asset volatility, $dB_v(t)$.

The instantaneous return on equity is computed via simple application of Itō's Lemma for Poisson processes:

$$\begin{aligned}
\frac{dE_t}{E_t} &= \Delta_t \frac{A_t}{D_t}\frac{D_t}{E_t} \cdot \frac{dA_t}{A_t} + \frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + \frac{1}{2E_t}\left[\frac{\partial^2 f}{\partial A_t^2} d\left\langle A\right\rangle_t + \frac{\partial^2 f}{d\left(\sigma_{A,t}\right)^2} d\left\langle \sigma_A^f\right\rangle_t + \frac{\partial^2 f}{\partial A \partial \sigma_{A,t}} d\left\langle A, \sigma_A\right\rangle_t\right] \\
&\quad + \left[\frac{E_t^J - E_t}{E_t}\right] dN_A(t)
\end{aligned} \tag{18}$$

where $\Delta_t = \partial f\left(A_t, D_t, \sigma_{A,t}, \tau, r\right)/\partial A_t$ is the "delta" in option pricing, $\nu_t = \partial f\left(A_t, D_t, \sigma_{A,t}, \tau, r\right)/\partial \sigma_{A,t}$ is the so-called "vega" of the option, and $\left\langle X\right\rangle_t$ denotes the quadratic variation process for an arbitrary stochastic process X_t. Additionally, E_t^J is the value of equity for an asset jump of $J_A = J$. Hence, E_t^J is itself a random variable. Once again, we have ignored the sensitivity of the option value to the maturity of the debt. All the quadratic variation terms are of the order $\mathcal{O}(dt)$ and henceforth we collapse them to an unspecified function $q(A_t, \sigma_{A,t}; f)$, where the notation captures the dependence of the higher order Itō terms on the partial derivatives of the call option pricing function.

The call option pricing function is still monotonically increasing in its first argument, so it is safe to assume that $f(\cdot)$ is invertible with respect to this argument. We further assume that the call pricing function is homogenous of degree one in its first two arguments. Define the inverse call option formula as follows:

$$\begin{aligned}
\frac{A_t}{D_t} &= g\left(E_t/D_t, 1, \sigma_{A,t}, J_A, N_A, \tau, r_t\right) \\
&\equiv f^{-1}\left(E_t/D_t, 1, \sigma_{A,t}, J_A, N_A, \tau, r_t\right)
\end{aligned} \tag{19}$$

Thus, Equation (18) reduces returns to the following:[25]

$$\begin{aligned}
\frac{dE_t}{E_t} &= \overbrace{\Delta_t \cdot g\left(E_t/D_t, 1, \sigma^f_{A,t}, \tau, r_t\right) \cdot \frac{D_t}{E_t}}^{\equiv LM\left(E_t/D_t, 1, \sigma_{A,t}, \tau, r_t\right)} \times \frac{dA_t}{A_t} + \frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + q(A_t, \sigma^f_{A,t}; f)dt + \left[\frac{E^J_t - E_t}{E_t}\right] dN_A(t) \\
&= LM\left(E_t/D_t, 1, \sigma_{A,t}, \tau, r_t\right) \times \frac{dA_t}{A_t} + \frac{\nu_t}{E_t} \cdot d\sigma_{A,t} + q(A_t, \sigma^f_{A,t}; f)dt + \left[\frac{E^J_t - E_t}{E_t}\right] dN_A(t)
\end{aligned} \tag{20}$$

Henceforth, when it is obvious, we will drop the functional dependence of the leverage multiplier on leverage, etc. and instead denote it simply by LM_t. In order to obtain a complete law of motion for equity, we use Itō's Lemma to derive the volatility process:

$$\begin{aligned}
d\sigma_{A,t} &= \overbrace{\left[\frac{\mu_v(t, v_t)}{2\sigma_{A,t}} - \frac{\sigma^2_v(t, v_t)}{8\sigma^3_{A,t}}\right]}^{\equiv s\left(\sigma_{A,t}; \mu_v, \sigma_v\right)} dt + \frac{\sigma_v(t, v_t)}{2\sigma_{A,t}} dB_v(t) \\
&= s\left(\sigma_{A,t}; \mu_v, \sigma_v\right) dt + \frac{\sigma_v(t, v_t)}{2\sigma_{A,t}} dB_v(t)
\end{aligned} \tag{21}$$

Plugging Equations (17), (21) into Equation (20) yields the desired full equation of motion for equity returns:

$$\begin{aligned}
\frac{dE_t}{E_t} &= \left[LM_t \mu_A(t) + s\left(\sigma_{A,t}; \mu_v, \sigma_v\right) + q(A_t, \sigma_{A,t}; f)\right] dt \\
&\quad + LM_t \sigma_{A,t} dB_A(t) + \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} dB_v(t) + \left[\frac{E^J_t - E_t}{E_t}\right] dN_A(t)
\end{aligned} \tag{22}$$

Since typical daily equity returns are virtually zero on average, we can continue to ignore the equity drift term. Instantaneous equity returns then naturally derive from Equation (22) with no drift:

$$\frac{dE_t}{E_t} = LM_t \sigma_{A,t} dB_A(t) + \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} dB_v(t) + \left[\frac{E^J_t - E_t}{E_t}\right] dN_A(t) \tag{23}$$

Our ultimate object of interest is the instantaneous volatility of equity, but in order to obtain a complete expression for equity volatility we have to determine the variance of the jump component of equity returns. In Appendix B, we derive an easily computed expression, denoted by $\mathcal{V}^J_A\left(J_A, N_A(t); A_t\right)$, that involves a simple integration over the normal density. Hence, total instantaneous equity volatility in this (reasonably) general setting is given by:

$$vol_t\left(\frac{dE_t}{E_t}\right) = \sqrt{LM^2_t \times \sigma^2_{A,t} + \frac{\nu^2_t \sigma^2_v(t, \sigma_{A.t})}{4E^2_t \sigma^2_{A,t}} + 2LM_t \sigma_{A,t} \frac{\nu_t}{E_t} \frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}} \rho_t + \mathcal{V}^J_A\left(J_A, N_A(t); A_t\right)} \tag{24}$$

There are four terms that contribute to equity volatility. The first term relates to asset volatility and the second relates to the volatility of asset volatility (as well as the sensitivity of the option to changes in volatility). The third depends on the correlation between assets innovations and asset volatility innovations. In practice, this correlation

[25] Using the fact that $f(\cdot)$ is homogenous of degree 1 in its first argument also implies that

$$\Delta_t = \partial f\left(A_t, D_t, \sigma_{A,t}, \tau, r\right)/\partial A_t = \partial f\left(A_t/D_t, 1, \sigma_{A,t}, \tau, r\right)/\partial(A_t/D_t)$$

So with an inverse option pricing formula, $g(\cdot)$ in hand we can define the delta in terms of leverage E_t/D_t.

is negative. Thus, the middle two terms in Equation (24) will have offsetting effects in terms of the contribution of stochastic asset volatility to equity volatility. Finally, the fourth term relates to the volatility of the jump process for assets. In later sections and also in the Appendix C, we show that we can ignore all but the first term for the purposes of volatility modeling in our context because, compared to asset volatility, they contribute very little to equity volatility. Thus, Equation (23) and (24) reduce to a very simple expression for equity returns and instantaneous volatility, and is the basis for Assumption 1:

$$\begin{aligned} \frac{dE_t}{E_t} &\approx LM_t \sigma_{A,t} dB_A(t) \\ vol_t\left(\frac{dE_t}{E_t}\right) &\approx LM_t \times \sigma_{A,t} \end{aligned} \tag{25}$$

Equation (25) is our key relationship of interest. It states that equity volatility (returns) is a scaled function of asset volatility (returns), where the function depends on financial leverage, D_t/E_t, as well as asset volatility over the life of the option (and the interest rate). The moniker of the "leverage multiplier" should be clear now. The functional form for $LM(\cdot)$ depends on a particular option pricing model, and one of our key contributions is to estimate a generalized $LM(\cdot)$ function that, in practice, encompasses a number of option pricing models. In order to build further intuition for the properties of the leverage multiplier, we examine it in three different option pricing models: Merton (1976), Heston (1993), and Bates (1996).

A.2 The Leverage Multiplier Under Various Option Pricing Models

The purpose of this subsection is twofold. We first aim to show that the approximation in Equation (25) performs well in a variety of different option pricing models. The main mechanism behind this approximation is as follows: as the time to maturity of the equity option gets larger, the contribution of the volatility of volatility (and jumps, if small enough in probability) to equity returns/volatility is minimal. This is primarily because volatility is mean reverting, and thus "long run" volatility is effectively constant, so that short run changes in volatility matter little for equity returns. In all of the option pricing models we consider, we assess the accuracy of this approximation in the same way. We simply parameterize the model and compute the total volatility of equity, given by Equation (24), within the model. Then, we compare it with the approximation of (25) within the same model. This comparison is conducted at debt maturities ranging from one month to twenty years. For each maturity, we choose the debt level such that the leverage is the same across maturities. For example, suppose we are examining Option Model A. For each maturity, we search for the debt (strike) such that debt to equity will be some arbitrary level L. For this debt level, we next compute the total equity volatility and the approximation under Option Model A; then we repeat this process for each maturity. We opt to keep leverage consistent across maturities since our empirical work will focus on financial firms whose leverage tends to be high, despite heterogenous debt maturities. The leverage level we target is $L = 15$. Why this number for leverage? For the empirical portion of our investigation, we examine financial firms. Financial firms typically have high levels of leverage ranging from 10-20, where leverage is measured as the book value of debt divided by the market value of equity. When possible, we are also careful to keep parameters consistent across all option pricing models we consider in order to keep the results comparable.

After we establish the validity of the approximation in Equation (25), we explore how the leverage multiplier behaves in a variety of option pricing settings. As we will show, Equation (25) is indeed a useful approximation, and the leverage multiplier takes on a similar shape across many different models.

A.2.1 The Heston (1993) Model

In order to justify Assumption 1, we study equity dynamics under the popular Heston (1993) stochastic volatility model. In this setting, asset volatility evolves as follows:

$$d\sigma_{A,t}^2 = \kappa\left[\theta - \sigma_{A,t}^2\right]dt + \eta\sigma_{A,t}dB_v(t)$$

The parameter κ determines how quickly volatility mean reverts, θ determines the asymptotic limit of volatility and η tunes the volatility of volatility.[26] This specification of asset returns can be viewed as a special case of Equation (2).

Validating Assumption 1 Our first task is to analyze how equity volatility is affected by this particular stochastic volatility model. In this model, it is straightforward from Equation (8) that the total volatility of equity is given by:

$$vol_t\left(\frac{dE_t}{E_t}\right) = \sqrt{LM_t^2 \times \sigma_{A,t}^2 + \frac{\nu_t^2\eta^2}{4E_t^2} + LM_t\sigma_{A,t}\frac{\eta\nu_t}{E_t}\rho_t\left(dB_A(t), dB_v(t)\right)} \tag{26}$$

Let us consider the magnitudes of each of the terms that contribute to equity volatility. The closed form option pricing formula in this setting was the major contribution of Heston (1993), and the greeks we are interested in do have closed forms, but we choose to compute them via numerical derivatives. For the remainder of this subsection, we assume that risk-neutral asset and volatility processes are specified by the Heston (1993) model, which we parametrize as follows:

Table 5: Heston (1993) Calibration for Risk-Neutral Asset Prices

Parameter	**Value**
κ	4
θ	0.15
η	0.15
$\sigma_{A,t}$	0.15
ρ	-0.5

Notes: This table provides the parameters we use to calibrate the risk-neutral asset return process under the Heston (1993) model.

Setting $\kappa = 4$ sets the half-life of the volatility process to be approximately 44 days, which is roughly the half-life of the GARCH processes we typically encounter in practice.[27] Setting $\theta = 0.15$ means that long-run asset volatility is 15% in annualized terms, which we adopt to keep consistency with the analysis in Section 2.2.1 (we also initialize current spot volatility to its long-run average). $\eta = 0.15$ makes the volatility of asset *volatility* to be 15% annually, or equal to the volatility of assets returns themselves. To the best of our knowledge, this does not contradict known fact.[28] Finally, we set the correlation between asset volatility shocks and asset return shocks to be -0.5 in order to capture the well-known asymmetry in volatility.

Our objective is to show that under the Heston (1993) specification, it is perfectly reasonable to ignore the terms in Equation (8) that are related to stochastic asset volatility. The argument for ignoring the stochastic portion of asset volatility in terms of its contribution to equity volatility hinges on the idea that for longer maturity debt, the volatility of asset volatility contributes very little to equity volatility. Thus, under the current parameterization of the Heston (1993) model, we plot the total instantaneous equity volatility against maturity in Equation (26) against maturity. In addition, we plot instantaneous equity volatility under Assumption 1.[29] Figure 12 provides a visualization of the results:

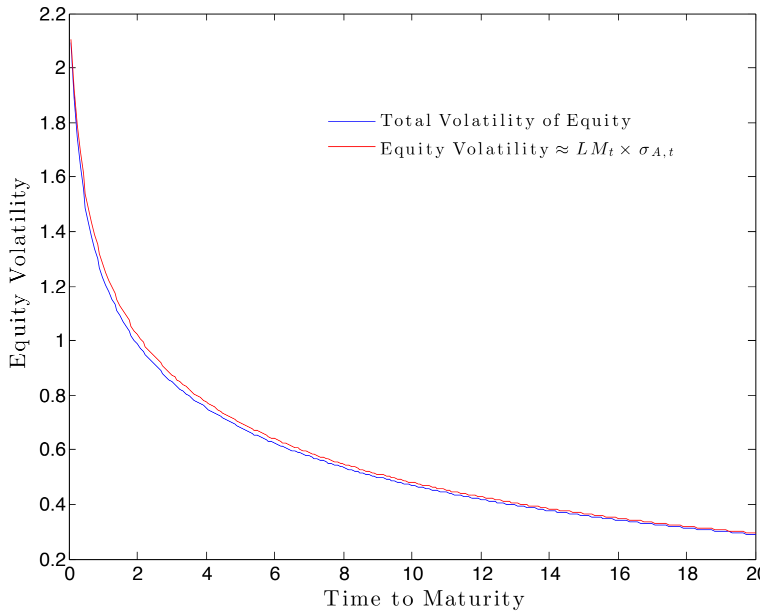
[26] For a more technical description of the properties of this volatility process see Heston (1993) or Cox, Ingersoll, and Ross (1985).

[27] The half-life of the volatility process is $ln(2)/\kappa$.

[28] In fact, this may even be too large. As assets are much less volatile than equity, so one might expect the volatility of asset volatility to be smaller than the volatility of equity volatility as well.

[29] For each maturity, we search for the debt (strike) such that debt to equity will be some arbitrary level L. For this debt level, we next compute the total equity volatility and the approximation under the Heston (1993) model; then we repeat this process for each maturity. We opt to keep leverage consistent across maturities since our empirical work will focus on financial firms whose leverage tends to be high, despite heterogenous debt maturities. The leverage level we target is $L = 15$. Why this number for leverage? For the empirical portion of our investigation, we examine financial firms. Financial firms typically have high levels of leverage ranging from 10-20, where leverage is measured as the book value of debt divided by the market value of equity.

Figure 12: Equity Volatility Analysis in Heston (1993) Model



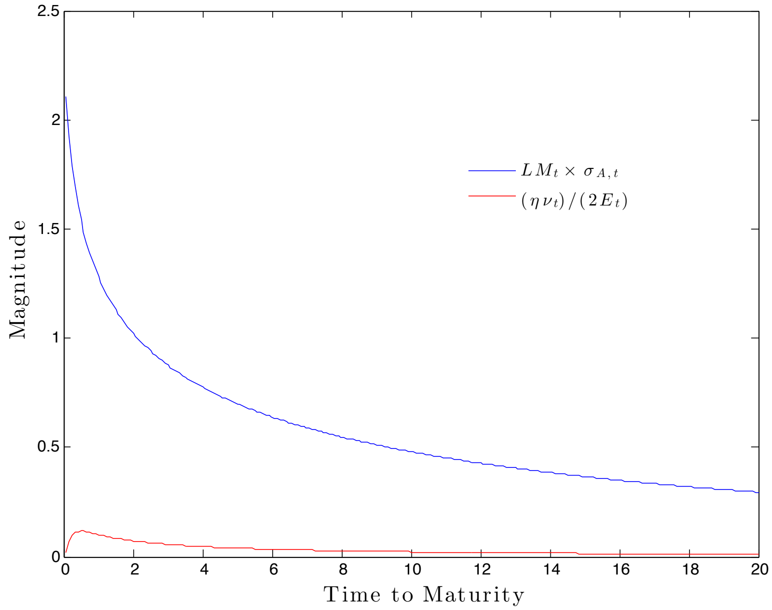
Notes: The blue line in this figure plots the total (true) volatility of equity if assets follow the Heston (1993) model. The red line in this figure plots the volatility of equity if it is approximated according to Equation (9). For each time to maturity, we set $A_t = 1$ and choose the level of debt such that financial leverage is approximately 15. For this analysis, the risk-free rate is set to 0.03.

As should be clear, the assumption underlying Equation (9) is quite reasonable in terms of capturing total equity volatility. The reason is due entirely to the mean-reversion property of asset volatility. Because asset volatility mean reverts quickly relative to the life of the debt, the "vega" of equity is very small. To see this point more forcefully, consider equity returns (with zero drift) if assets follows the Heston model:

$$\frac{dE_t}{E_t} = LM_t \sigma_{A,t} dB_A(t) + \frac{\eta \nu_t}{2E_t} dB_v(t)$$

How big is each term multiplying the shocks in this model? Staying with the parameters in Table 5, we plot both below:

Figure 13: Equity Return Analysis in Heston (1993) Model



Notes: The blue line in this figure plots the how much asset return shocks are magnified into equity shocks. The red line in this figure plots how much volatility shocks are amplified into equity shocks. For each time to maturity, we set $A_t = 1$ and choose the level of debt such that financial leverage is approximately 15. For this analysis, the risk-free rate is set to zero.

Clearly, the contribution of volatility shocks to equity returns is minimal. This result shouldn't be surprising as it is essentially a restatement of the conclusion reached from Figure 12.[30]

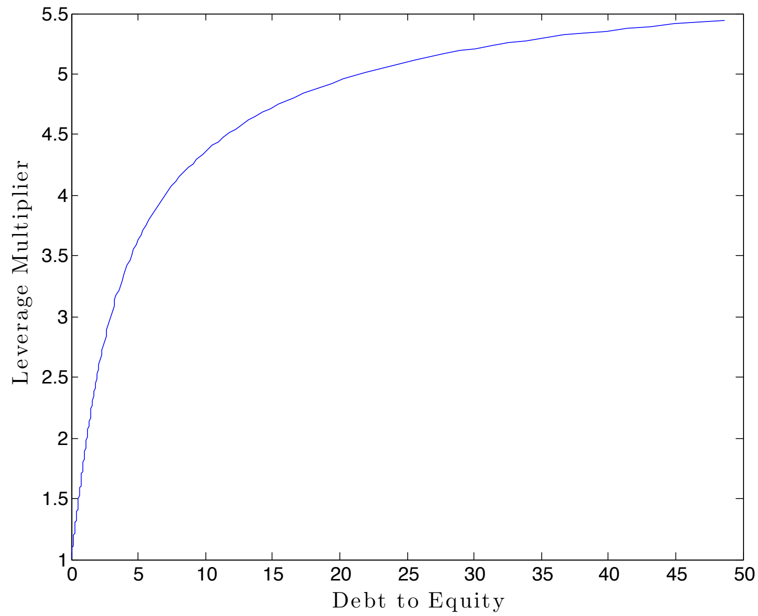
The Leverage Multiplier in the Heston (1993) Model Now that we have established that the leverage multiplier is the key determinant of equity dynamics in this stochastic volatility environment, we turn to a more careful analysis of the actual shape of the leverage multiplier. Figure 14 plots the leverage multiplier (for debt of maturity $\tau = 5$) against leverage as we did in the BSM case:

[30]Hence, we arrive at the previously assumed process for equity returns, even in a stochastic volatility environment:

$$\frac{dE_t}{E_t} = LM_t \times \sigma_{A,t} dB_A(t)$$

This isn't to say that the volatility of volatility terms do not matter. They matter quite a bit for the *level* of equity, but not so much for the returns.

Figure 14: Leverage Multiplier in Heston (1993) Model



Notes: We plot the leverage multiplier in the Heston (1993) stochastic volatility model against leverage. The time to maturity is set to five years and the remaining model parameters are given in Table 5. For this analysis, the risk-free rate is set to $r = 0.03$.

The shape of the leverage multiplier under stochastic volatility is quite similar to the BSM case. The most obvious difference is the concavity of the stochastic volatility leverage multiplier. When moving from low levels of leverage (e.g. 2-3) to intermediate levels of leverage (e.g. 5-7) the leverage multiplier rises rapidly as compared to the BSM baseline. In addition, the level of the stochastic volatility leverage multiplier is higher than the BSM counterpart. Because of the negative correlation between asset returns and asset volatility, the future asset return distribution has negative skewness. In turn, leverage has strong effects on equity returns and volatility since the option value of equity is less likely to expire in the money. This intuition is also consistent with the results of section 2.2.3, where we use Monte Carlo simulation to explore models with stochastic volatility and non-normal shocks.

A.2.2 Analysis of the Leverage Multiplier under Merton's (1976) Jump-Diffusion Model

Another strand of option pricing models in the literature began with the seminal work of Merton (1976), henceforth MJD. In this formulation, the stock returns follow a standard geometric Brownian motion, appended with a continuous time Poisson jump process. Again, this case is encompassed by the specification in Equation (23), but we turn off the stochastic volatility channel (i.e. $\sigma_{A,t} = \sigma_A$) and allow only for jumps. In this setting, equity return dynamics from (23) reduce to the following jump-diffusion:

$$\frac{dE_t}{E_t} = LM_t \sigma_A dB_A(t) + \left[\frac{E_t^J - E_t}{E_t}\right] dN_A(t)$$

Similarly, the volatility of equity returns reduces to the following:

$$vol_t\left(\frac{dE_t}{E_t}\right) = \sqrt{LM_t^2 \times \sigma_A^2 + \mathcal{V}_A^J\left(J_A, N_A(t); A_t\right)}$$

where again the expression for $\mathcal{V}_A^J(J_A, N_A(t); A_t)$ is found in Appendix B. The key insight of MJD model was that, even though dynamic riskless hedging is impossible with discontinuous sample paths, if individual stock jumps are independent of the prevailing pricing kernel then their presence is an "unpriced" risk and typical hedging arguments can still be applied. Merton (1976) solves for the closed form option pricing solution, which turns out to be a weighted average of BSM prices, with the weights determined by the likelihood of a given number of jumps over the life of the debt.[31] Our analysis of his solution begins with an assessment of the approximation in Equation (25).

Validating Assumption 1 In order to do so, we first parameterize the MJD model as follow:

Table 6: Merton (1976) Calibration for Risk-Neutral Asset Prices

Parameter	**Value**
λ	0.01
μ_J	-0.1
σ_J	0.2

Notes: This table provides the parameters we use to calibrate the asset return process under the Merton (1976) model.

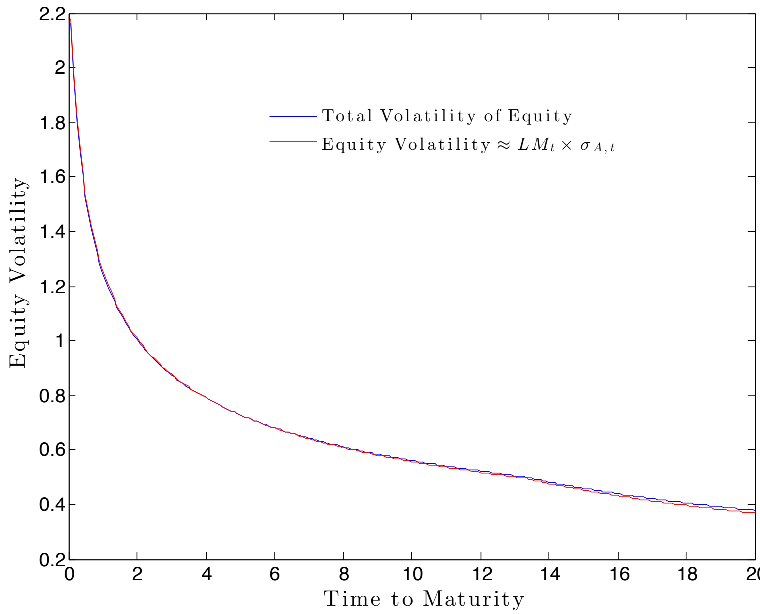
In this case, the average jump size (μ_J) means that when a jump occurs, the asset value falls by 10%, and has a dispersion (σ_J) of 20%. Jumps happen at an annualized frequency of λ, which means there are roughly two expected days per year with jumps for assets returns. In addition, it is well known that the total instantaneous volatility of assets is given by:

$$vol_t\left(\frac{dA_t}{A_t}\right) = \sqrt{\sigma_A^2 + \lambda\left[\mu_J^2 + \left(e^{\sigma_J^2} - 1\right)(1+\mu_J)^2\right]}$$

Thus, in order to keep the analysis comparable to the baseline BSM case, we set $\sigma_{A,t} = \sigma_A$ such that the total annualized asset volatility is 15%. Using only the leverage multiplier to compute equity volatility (i.e. ignoring volatility added by jumps) is visualized in the following plot:

[31] To use the exact Merton (1976) formula in our context, set the mean of the "Merton Jump", which is log-normal to $\mu_M = \log(1+\mu_J) - \sigma_J^2/2$ in the specification we outline for jumps. The variance is the same.

Figure 15: Accuracy of Volatility Approximation in Merton (1976) Model

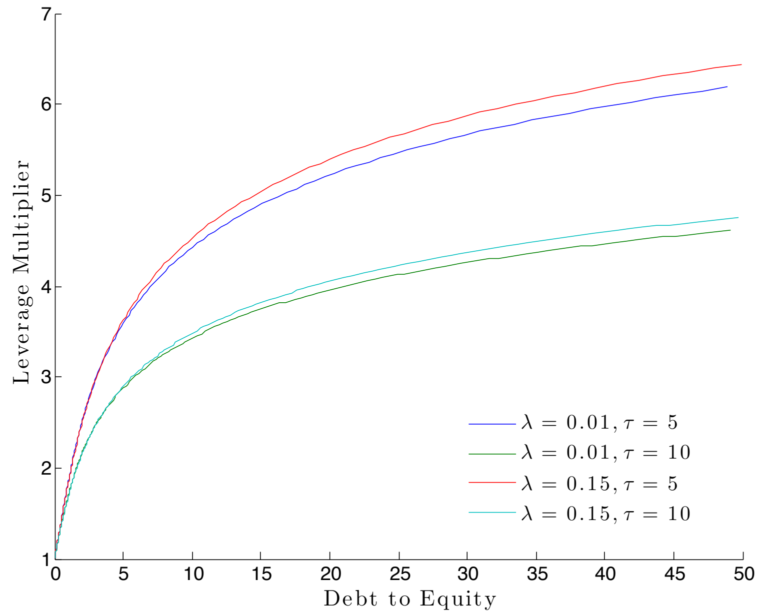


As is evident from Figure 15, adding jumps to the asset return process has a very small effect on overall equity volatility. The variance of the entire jump process, $J_A dN_A(t)$, is largely dictated by the variance of the Poisson counting process, which is λ. Therefore, for reasonable jump arrival intensities, the variance contribution of jumps to equity volatility will be small in magnitude.[32] Now that we have established that approximating equity volatility by Equation (25), we turn our attention to understanding the properties of the leverage multiplier in this context.

The Leverage Multiplier in the Merton (1976) Model We study the properties of the Merton (1976) leverage multiplier by varying the jump intensity λ. Again, in order to make these results comparable to the benchmark BSM analysis, we always set σ_A such that the annualized total volatility of assets is 15%. In addition, we vary the time to maturity of the debt. The leverage multiplier for these different cases is as follows:

[32] In the Online Appendix, we repeat this analysis for other parameterizations and find the conclusions to be robust.

Figure 16: Leverage Multiplier in Merton (1976) Jump-Diffusion Model



Note: This figure plots the leverage multiplier in the MJD model. Annualized jump intensity takes on one of two values $\lambda \in \{0.01, 0.15\}$. The time to maturity of the debt also takes on two possible values $\tau \in \{5, 10\}$. The annualized risk-free rate is $r = 0.03$.

The economic underpinnings of the leverage multiplier in the MJD case are, unsurprisingly, quite similar to the BSM case. Holding time to maturity constant, when we decrease the likelihood of a jump, the leverage multiplier also decreases. Since jumps are assumed to decrease the asset value, then a lower likelihood of a jump means the equity is more likely to finish in the money and the effect of leverage on equity volatility is dampened. Holding the likelihood of a jump constant, when we increase the time to maturity, the leverage multiplier increases. In this case, the diffusion portion of assets dominates the negative jump component (due to the parameter values we chose). Adding maturity to the debt therefore means the equity is more likely to expire with value and so leverage means less for equity volatility. It is likely that for large jump intensities (or large jump sizes) this effect would reverse, but this type of parameterization seems empirically implausible.

A.2.3 Analysis of the Leverage Multiplier with Stochastic Volatility and Jumps

By now it should be clear that the leverage multiplier takes roughly the same form across different option pricing models; however, for completeness, we conduct one last exploration of the leverage multiplier shape when assets have both stochastic volatility and are subject to jumps (henceforth SVJ). The risk-neutral asset return dynamics are thus described as follows:

$$\begin{aligned}\frac{dA_t}{A_t} &= [r - \lambda\mu_J]dt + \sigma_{A,t}dB_A(t) + JdN_A(t)\\ d\sigma^2_{A,t} &= \kappa\left[\theta - \sigma^2_{A,t}\right]dt + \eta\sigma_{A,t}dB_v(t)\end{aligned}$$

where the jump process for assets retains its original properties as in Equation (17). The closed form solution for option prices under these dynamics is also well-known at this point (e.g. Bakshi, et al. (1997)). We calibrate the model by combining previous parameterizations and repeat them here:

Table 7: Parameters for Stochastic Volatility with Jumps Model

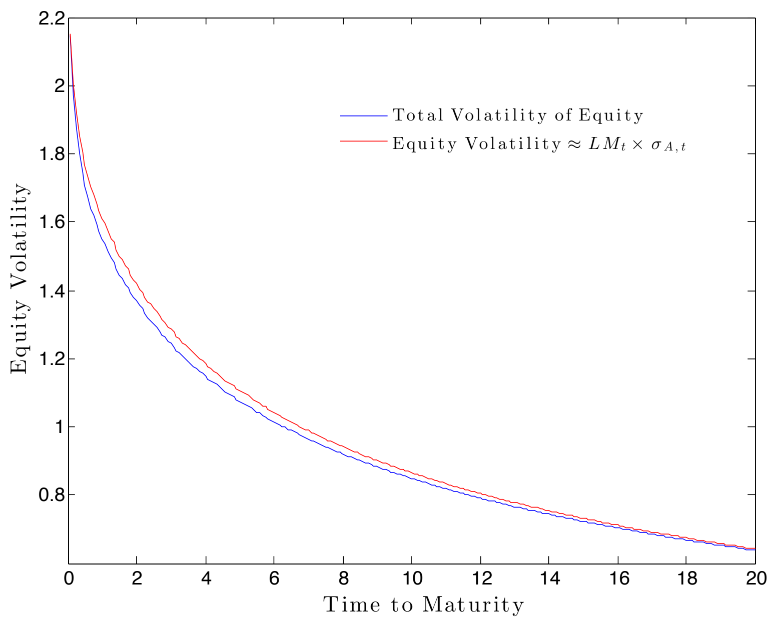
Parameter	Value
κ	4
η	0.15
$\sigma_{A,t}$	0.15
ρ	-0.5
λ	0.01
μ_J	-0.1
σ_J	0.2

Notes: This table provides the parameters we use to calibrate the risk-neutral asset return process under the SVJ model used by, among others, Bates (1996) and Bakshi et al. (1997).

The long run average for volatility, θ, set such that the total long run average asset volatility is 15%. As usual, we begin by checking whether the approximation in Equation (25) holds in this setting.

Validating Assumption 1 Figure 17 plots the total equity volatility in the SVJ model against the approximate equity volatility given by (25) for differing maturities and constant leverage:

Figure 17: Equity Volatility Analysis in SVJ Model

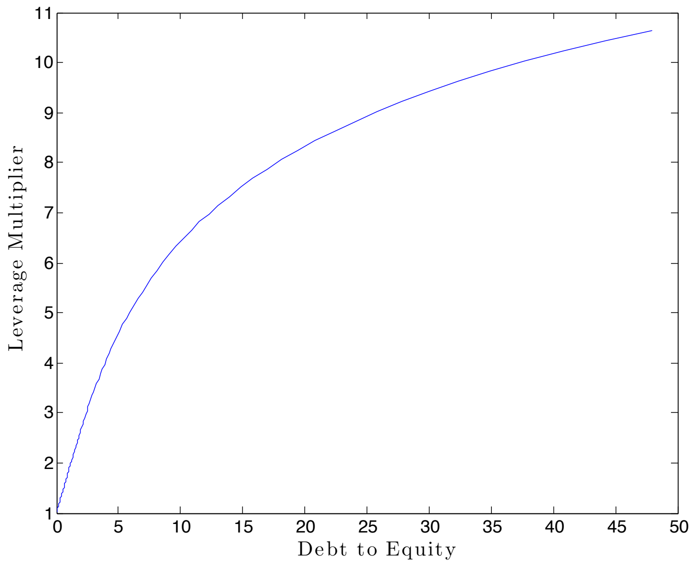


Notes: The blue line in this figure plots the total (true) volatility of equity if assets follow the SVJ model (e.g. Bakshi et al. (1997)). The red line in this figure plots the volatility of equity if it is approximated according to Equation (25). For each time to maturity, we set $A_t = 1$ and choose the level of debt such that financial leverage is approximately 15. For this analysis, the risk-free rate is set to 0.03.

At this point, it should not be so surprising that our approximation holds reasonably well. Simply put, for reasonable jump arrival intensities and quickly mean-reverting volatility processes, the main component of equity volatility is asset volatility itself (amplified by the leverage multiplier).

The Leverage Multiplier in the SVJ Model Similarly, Figure 18 plots the leverage multiplier in the SVJ model:

Figure 18: Leverage Multiplier in SVJ Model



Notes: We plot the leverage multiplier in the SVJ model against leverage. The time to maturity is set to five years and the remaining model parameters are given in Table **??**. For this analysis, the risk-free rate is set to $r = 0.03$.

As expected, the leverage multiplier in the SVJ cases is the highest for high levels of leverage. The intuition from the BSM model still applies: high leverage means the equity is very likely to expire out of the money for a couple of reasons. First, because volatility and asset return shocks are negatively correlated, the distribution of future asset returns has negative skewness, thus making high levels of leverage even more paralyzing in terms of equity finishing with value. The second reason is due to jumps, which we assume to be negative on average. Together, both factors contribute to a higher equity volatility amplification mechanism when leverage becomes too high. Still, the general shape of the leverage multiplier seems consistent across all of the option pricing models, and we therefore confirm that the results from the main text remain valid.

B Appendix: Total Volatility of Equity in Stochastic Volatility with Jump Environment

We need to compute the variance of the following term:

$$\begin{aligned} \mathcal{V}_A^J\left(J_A, N_A(t); E_t\right) &\equiv var_t\left(\left[\frac{E_t^J - E_t}{E_t}\right] dN_A(t)\right) \\ &= var_t\left(\frac{E_t^J}{E_t} dN_A(t)\right) + var_t\left(dN_A(t)\right) \end{aligned}$$

where we use the independence of $N_A(t)$ and J_A. To make the dependence of equity value on the asset jump more explicit, we replace E_t^J with the call option pricing function, but we suppress all but the first argument for notional convenience:

$$E_t^J = f\left(A_t + J_A\right)$$

Using the definition of variance and the properties of the Poisson counting process (i.e. variance is λ), we get the following expression for $\mathcal{V}_A^J(J_A, N_A(t); E_t)$:

$$\begin{aligned}
\mathcal{V}_A^J(J_A, N_A(t); A_t) &= \frac{1}{A_t^2}\left\{\mathbb{E}\left[f(A_t+J_A)^2 dN(t)^2\right] - \mathbb{E}\left[f(A_t+J_A)dN(t)\right]^2\right\} + \lambda \\
&= \frac{1}{A_t^2}\left\{\mathbb{E}\left[f(A_t+J_A)^2\right]\mathbb{E}\left[dN(t)^2\right] - \mathbb{E}\left[f(A_t+J_A)\right]^2\mathbb{E}\left[dN(t)\right]^2\right\} + \lambda \\
&= \frac{1}{A_t^2}\left\{\mathbb{E}\left[f(A_t+J_A)^2\right]\left(\lambda+\lambda^2\right) - \mathbb{E}\left[f(A_t+J_A)\right]^2\lambda^2\right\} + \lambda \\
&= \frac{1}{A_t^2}\left\{\psi(A_t, J_A)\left(\lambda+\lambda^2\right) - \Gamma(A_t, J_A)\lambda^2\right\} + \lambda
\end{aligned}$$

Here, the second line uses the independence of $N_A(t)$ and J_A, and the third line uses the standard variance definition for a Poission process.[33] $\psi(\cdot)$ and $\Gamma(\cdot)$ are given by:

$$\begin{aligned}
\psi(A_t, J_A) &= \mathbb{E}\left[f(A_t+J_A)^2\right] \\
&= \int_{-\infty}^{\infty} f\left(A_t+e^y-1\right)^2 h_Y(y)dy \\
\Gamma(E_t, J_A) &= \left[\int_{-\infty}^{\infty} f\left(A_t+e^y-1\right) h_Y(y)dy\right]^2
\end{aligned}$$

where $h_Y(y)$ is the pdf of a normal random variable with mean $\log[1+\mu_J] - \sigma_J^2/2$ and variance σ_J^2. In practice, $\mathcal{V}_A^J(J_A, N_A(t); A_t)$ is easily computed numerically.

C Appendix: Empirical Argument for Ignoring Volatility Terms

For exposition, we repeat Equation (3):

$$\frac{dE_t}{E_t} = \Delta_t \cdot \frac{A_t}{D_t} \cdot \frac{D_t}{E_t} \cdot \frac{dA_t}{A_t} + \frac{\partial f}{\partial \sigma_{A,t}^f} \cdot \frac{D_t}{E_t} \cdot d\sigma_{A,t}^f \tag{27}$$

It is straightforward to work out that equity variance will be as follows:

$$\begin{aligned}
var_t\left(\frac{dE_t}{E_t}\right) &= \left(\Delta_t \frac{A_t}{Dt} \cdot \frac{D_t}{E_t}\right)^2 var_t\left(\frac{dA_t}{A_t}\right) + \left(\frac{\nu_t D_t}{E_t}\right)^2 var_t\left(d\sigma_{A,t}^f\right) \\
&\quad +2\left(\Delta_t \frac{A_t}{Dt} \cdot \frac{D_t}{E_t}\right)\left(\frac{\nu_t D_t}{E_t}\right)\sqrt{var_t\left(\frac{dA_t}{A_t}\right) var_t\left(d\sigma_{A,t}^f\right)} \times \rho_t\left(\frac{dA_t}{A_t}, d\sigma_{A,t}^f\right)
\end{aligned} \tag{28}$$

[33] i.e.

$$\begin{aligned}
var_t(dN_A(t)) &= \mathbb{E}\left[dN_A(t)^2\right] - \mathbb{E}\left[dN_A(t)\right]^2 \\
&\Leftrightarrow \\
\mathbb{E}\left[dN_A(t)^2\right] &= var_t(dN_A(t)) + \mathbb{E}\left[dN_A(t)\right]^2 \\
&= \lambda + \lambda^2
\end{aligned}$$

where $\Delta_t = \partial f\left(A_t/D_t, 1, \sigma^f_{A,t}, \tau, r\right)/\partial A_t$ and $\nu_t = \partial f\left(A_t/D_t, 1, \sigma^f_{A,t}, \tau, r\right)/\partial d\sigma^f_{A,t}$. In option pricing lingo, Δ_t is the "delta" of the call option on assets, and ν_t is the "vega" of the call option on assets.[34] In the model where we set the long run volatility of assets to be the unconditional volatility of the asset GJR process, this analysis is moot as $d\sigma^f_{A,t} = 0$. Our task now is to show that the last two terms are negligible for the purposes of modeling equity volatility, when we use the GJR forecast for long run asset volatility.

C.1 Magnitude of Volatility Terms

In the language of the Structural GARCH model we can simply substitute LM_t into Equation (28) where it is appropriate:

$$\begin{aligned} var_t\left(\frac{dE_t}{E_t}\right) &= (LM_t)^2\, var_t\left(\frac{dA_t}{A_t}\right) + \left(\frac{\nu_t D_t}{E_t}\right)^2 var_t\left(d\sigma^f_{A,t}\right) \\ &\quad +2\left(LM_t\right)\left(\frac{\nu_t D_t}{E_t}\right)\sqrt{var_t\left(\frac{dA_t}{A_t}\right) var_t\left(d\sigma^f_{A,t}\right)} \times \rho_t\left(\frac{dA_t}{A_t}, d\sigma^f_{A,t}\right) \end{aligned} \tag{29}$$

In order to investigate the magnitude of the terms we ignore (i.e. any term containing ν_t), we need a functional form for the sensitivity of the equity value to changes in long run asset volatility. Since we are only interested in magnitudes, we will use the Black-Scholes vega. It is unlikely that the Black-Scholes vega is incorrect by an order of magnitude, so for this exercise it will be sufficient. The next thing we need in order to quantitatively evaluate Equation (29) are time-series for LM_t, dA_t/A_t, and $d\sigma^f_{A,t}$. To be precise, if we extended the model to include changes in volatility we would undoubtedly obtain different estimates for these three quantities. Again, since our goal is to assess relative magnitudes, we will simply use the values delivered by our Structural GARCH model for LM_t, dA_t/A_t, and $d\sigma^f_{A,t}$. Formally, we define $d\sigma^f_{A,t}$ as:

$$d\sigma^f_{A,t} = \sqrt{h^f_{A,t+1}/\tau} - \sqrt{h^f_{A,t}/\tau}$$

where $h^f_{A,t+1}$ is the forecast of total volatility over the life of the option. Finally, we set the volatility of volatility to be constant and the correlation between asset returns and volatility to be constant. Under this assumption, we estimate these quantities using their in-sample moments.

C.1.1 Case Study: JPM

As an example, we study JPM. Table 8 are the in-sample moments of the estimated stochastic volatility process:

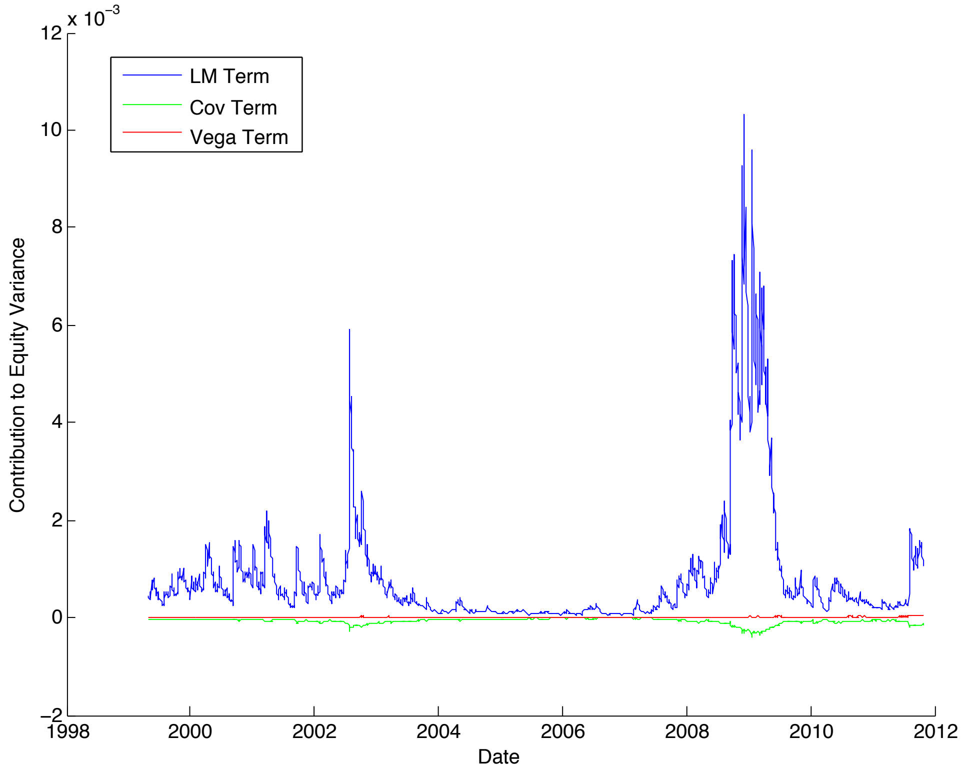
Table 8: Moments for Volatility Forecast Innovations

Variable	Value
$\sqrt{var_t\left(d\sigma^f_{A,t}\right)}$	4.9737e-04
$\rho_t\left(\frac{dA_t}{A_t}, d\sigma^f_{A,t}\right)$	-0.427

Unsurprisingly, there is a strong negative correlation between the innovation to the long run asset volatility forecast and asset returns. In addition, the extremely small volatility of volatility provides us our first piece of supporting evidence in favor of ignoring the additional vega terms. Next, we plot each of the three terms from Equation (29):

[34] To be precise, these are the delta and vega of the option where debt has been normalized to 1.

Figure 19: Decomposition of Equity Variance for JPM



Notes: The figure above plots each time series for each of the three terms in Equation (29). The blue line represents the first term, containing LM_t^2. The red line is the second term, and contains only $var_t(d\sigma_{A,t}^f)$. Finally, the green line is last covariance term.

Figure 19 confirms our assumption that including $d\sigma_{A,t}^f$ has a small effect on our main volatility specification. Moreover, it is clear from Equation (29) that any additional volatility of volatility that contributes to equity volatility will be offset by the negative correlation between asset volatility and asset returns. On average, the variance arising from our standard leverage multiplier term is approximately 12 times larger (in absolute value) than the sum the additional terms due to vega. Furthermore, the leverage multiplier term dominates the vega terms in times of high volatility, which tend to be our main areas of interest. Thus, we conclude that we can ignore the additional vega terms within the context of our Structural GARCH model.

D Data Appendix

D.1 List of Firms Analyzed

Table 9: List of Firms Analyzed

ABK	CBSS	HRB	SAF
ACAS	CFC	HUM	SEIC
AET	CI	JNS	SLM
AFL	CIT	JPM	SNV
AGE	CMA	KEY	SOV
AIG	CNA	LEH	STI
ALL	COF	LNC	STT
AMP	CVH	MBI	TMK
AMTD	ETFC	MER	TROW
AON	ETN	MET	TRV
AXP	EV	MI	UB
BAC	FITB	MMC	UNM
BBT	FNF	MS	USB
BEN	FNM	MTB	WB
BK	FRE	NCC	WFC
BLK	GNW	PBCT	WM
BOT	GS	PFG	WRB
BSC	HBAN	PGR	ZION
C	HCBK	PNC	
CB	HIG	PRU	
CBH	HNT	RF	