

Taming the Factor Zoo*

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Abstract

The asset pricing literature has produced hundreds of potential risk factors. Organizing this “zoo of factors” and distinguishing between useful, useless, and redundant factors require econometric techniques that can deal with the curse of dimensionality. We propose a model-selection method to systematically evaluate the contribution to asset pricing of any new factor, above and beyond what a high-dimensional set of existing factors explains. Our methodology explicitly accounts for potential *model-selection mistakes*, unlike the standard approaches that assume perfect variable selection, which rarely occurs in finite sample and produces a bias due to the omitted variables. We apply our procedure to a set of factors recently discovered in the literature. We show that several factors – such as profitability and investments – have statistically significant explanatory power beyond the hundreds of factors proposed in the past. In addition, we show that our risk price estimates and their significance are stable, whereas the model selected by simple LASSO is not.

Key words: Factors, Risk Price, Post-Selection Inference, Regularized Two-Pass Estimation, Variable Selection, Machine Learning, LASSO, Elastic Net.

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1 Introduction

The search for factors that explain the cross section of expected stock returns has produced hundreds of potential factor candidates, as noted by [Cochrane \(2011\)](#) and most recently by [Harvey et al. \(2015\)](#), [McLean and Pontiff \(2016\)](#), and [Hou et al. \(2016\)](#). A fundamental task facing the asset pricing field today is to bring more discipline to this “zoo” of factors. In particular, how can we discriminate truly useful pricing factors from redundant and useless factors that appear significant due to data mining? How do we judge whether a new factor adds explanatory power for asset pricing, relative to the existing set of hundreds of factors the literature has so far produced?

This paper provides a framework for systematically evaluating the contribution of individual factors relative to the myriad of existing factors the literature has proposed, and conducting appropriate statistical inference in this high-dimensional setting. In particular, we show how to estimate and test the marginal importance of any factor g_t in pricing the cross section of expected returns *beyond* what is explained by a high-dimensional set of potential factors h_t , where g_t and h_t could be tradable or non-tradable factors. We assume the true asset pricing model is approximately low-dimensional; however, in addition to relevant asset pricing factors, g_t and h_t include redundant ones that add no explanatory power to the other factors, as well as useless ones that have no explanatory power at all. Selecting the relevant factors from h_t and conducting proper inference on the contribution of g_t above and beyond those factors is the aim of this paper.

When h_t consists of a small number of factors, testing whether g_t is useful in explaining asset prices while controlling for the factors in h_t is straightforward: it simply requires estimating the loadings of the stochastic discount factor on g_t and h_t (i.e., the *price of risk* of these factors), and testing whether the price of risk of g_t is different from zero (see [Cochrane \(2009\)](#)). This exercise not only tells us whether g_t is useful for pricing the cross section, but it also reveals how shocks to g_t affect marginal utility, which has a direct economic interpretation.

When h_t consists of potentially hundreds of factors, however, standard statistical methods to estimate and test risk prices become infeasible or result in poor estimates and invalid inference, because of the curse of dimensionality. Although dimension-reduction techniques (e.g., least absolute shrinkage and selection operator, LASSO) can be useful in (asymptotically) selecting the right model and reducing the dimensionality of h_t , they produce *erroneous inference* unless appropriate econometric methods are used to explicitly account for the model-selection mistakes that can occur in any finite sample (see [Chernozhukov et al. \(2015\)](#)). For example, simply applying a model-selection tool like LASSO to a large set of factors and checking whether a particular factor g_t is significant in that sample (or even just checking if it gets selected) is not a reliable way to determine whether g_t is actually one of the true factors.

The methodology we propose in this paper marries these new econometric methods (in particular, the *double-selection LASSO* method of Belloni et al. (2014b)) with two-pass regressions such as Fama-MacBeth to specifically estimate risk prices in a high-dimensional setting and evaluate the contribution of a factor to explaining asset prices. Without relying on prior knowledge about which factors to include as controls among the hundreds in h_t , our procedure selects factors that are *either* useful in explaining the cross section of expected returns *or* are useful in mitigating the omitted variable bias problem due to potential model selection mistakes. We show that including both types of factors as controls is essential to conduct correct inference on the price of risk of g_t .

We apply our methodology to a large set of factors proposed in the last 30 years. We collect and construct a factor data library, containing 99 risk factors. Additionally, we create nontradable and nonlinear quadratic factors, including the squares of each of these 99 factors and their interactions with the size factor, respectively. We perform a variety of empirical exercises that illustrate the importance of accounting for model-selection mistakes when conducting inference about risk prices and assessing the significance of new factors. We start by evaluating the marginal contribution of recent factors proposed in the last five years to the large set of factors proposed before 2011. The new factors include – among others – the two new factors introduced by Fama and French (2015) and Hou et al. (2014), and the intermediary-based factors from He et al. (2016). Given that the set of potential control factors includes nearly 250 factors, one might wonder whether, in practice, any additional factor could make a significant contribution to explaining the cross section of expected returns. We show that indeed several of these new factors (e.g., profitability and investment) have significant marginal explanatory power for expected returns.

Second, we show that the results of this empirical exercise are stable across both the cross section of test assets and time periods. We randomly select 2,000 subsamples with replacement from our full sample, resampling in both the cross-sectional and time-series dimensions, and show that our estimates of risk prices and their significance are stable across subsamples. In contrast, the factors selected by simple LASSO in each subsample are *not* stable across subsamples. This result underscores the fact that in finite samples LASSO will generally pick some wrong factors; therefore, simple LASSO should not be used to determine whether any particular factor g_t belongs to the true model. Our double-selection procedure corrects for the potential mistakes by the LASSO selection thanks to a second selection step, described in greater detail below, and produces valid inference about the factors of interest despite the mistakes that occur at the factor-selection stage.

Third, we propose a recursive exercise in which factors are tested as they are introduced against previously proposed factors. The exercise shows that our procedure would have deemed factors as redundant or spurious in most cases, while finding significance for 14 of the factors (out of 99). Over time, therefore, our procedure would have screened out many factors at the time of their introduction.

The double-selection (DS) estimation procedure we propose, that combines cross-sectional asset pricing regressions with the double-selection LASSO of [Belloni et al. \(2014b\)](#) (designed originally for linear treatment effect models), starts by using a two-step selection method to select “control” factors from h_t , and then estimates the risk price of g_t from cross-sectional regressions that include g_t and the selected factors from h_t .

As the name implies, the “double selection” of factors from h_t happens in two stages; both stages are crucial to obtain correct inference on g_t . A first set of factors is selected from h_t based on their pricing ability for the cross-section of returns. Factors that appear to contribute little to pricing assets are excluded from the set of controls. This first step – effectively an application of standard LASSO to the set of potential factors h_t – has the advantage of selecting factors based on their usefulness in pricing the cross section of assets, as opposed to other commonly used selection methods (e.g., principal components) that select factors based on their ability to explain the time-series variation of returns. Using a cross-sectional approach is expected to deliver more relevant factors for asset pricing.

This first step is, however, not sufficient to ensure valid inference on g_t , because the LASSO selection may exclude some factors that have small risk prices in sample, but whose covariance with returns are nonetheless highly cross-sectionally correlated with exposures to g_t . That is, in any finite sample, we can never be sure that LASSO has selected the correct control model. Any omission of relevant factors due to finite-sample model-selection errors distorts the inference on the risk price of g_t , leading to incorrect inference on the significance – and even the sign – of g_t ’s risk price. This issue is a well-known problem with model-selection methods (see, for example, [Leeb and Pötscher \(2005\)](#)), and it has spurred a large econometrics literature on uniformly valid inference, with important consequences for asset pricing tests that we explore in this paper.

To obtain correct asymptotic inference for g_t , instead, including a second stage of factor selection is crucial. The second step adds to the set of controls additional factors whose covariances with returns are highly correlated in the cross section with the covariance between returns and g_t . Intuitively, we want to make sure to include even factors with small in-sample risk prices, if omitting them may still induce a large omitted variable bias due to the cross-sectional correlation between their risk exposures and the risk exposures to g_t . It is also possible that some variables selected from the second stage are useless or redundant, but their inclusion only leads to a moderate loss in efficiency.

After selecting the set of controls from h_t (including all factors selected in either of the two selection stages), we conduct inference on g_t by estimating the coefficient of a standard two-pass regression using g_t and the selected small number of control factors from h_t . This post-selection estimation step is also useful to remove biases arising from regularization in any LASSO procedure;

see, for example, [Friedman et al. \(2009\)](#). We then conduct asymptotic inference on the risk price of g_t using a central-limit result we derive in this paper. We show by simulation that our estimator performs well in finite samples, and substantially outperforms alternative estimators.

Finally, it is worth pointing out an alternative motivation for the methodology proposed in this paper. Theoretical asset pricing models often predict that some factors (g_t) should be part of the stochastic discount factor, i.e. they should enter the investors' marginal utility. Theoretical models, however, are often very stylized, and their ability to explain the cross-section is limited. This suggests that, in reality, investors may care about other risk factors that are not explicitly predicted by the model. This creates an omitted variable problem when testing for the risk price of g_t : if the true stochastic discount factor contains additional factors that are not explicitly incorporated in the estimation, the estimate for the price of risk of g_t will be biased. Our methodology – that estimates the price of risk of g_t while taking a stand on the “omitted factors” by choosing them from the large set h_t – can then be seen as a way to address this omitted factor concern when estimating risk prices. In this sense, it is related to [Giglio and Xiu \(2016\)](#), that show how to make inference on risk premia in the presence of omitted factors. The crucial difference between the two approaches is that [Giglio and Xiu \(2016\)](#) focus on the estimation of risk *premia* (the compensation investors require for holding the g_t risk), whereas this paper makes inference on risk *prices* of observable factors g_t (the coefficient of g_t in the stochastic discount factor). Both risk prices and risk premia have important, though very distinct, economic interpretations; they have different theoretical properties, and different tools need to be used to address the omitted factor problem in the two cases. Importantly, only risk prices, addressed in this paper, can speak to the contribution of factors to explaining asset prices (see [Cochrane \(2009\)](#)), and therefore risk prices are the appropriate concept to refer to for disciplining the zoo of factors.

Our paper builds on several strands of the asset pricing and econometrics literature. First and most directly, the paper is related to the recent literature on the high dimensionality of cross-sectional asset pricing models. [Green et al. \(2016\)](#) test 94 firm characteristics through Fama-Macbeth regressions and find that 8-12 characteristics are significant independent determinants of average returns. [McLean and Pontiff \(2016\)](#) use an out-of-sample approach to study the post-publication bias of 97 discovered risk anomalies. [Harvey et al. \(2015\)](#) adopt a multiple testing framework to re-evaluate past research and suggest a new benchmark for current and future factor fishing. Following on this multiple-testing issue, [Harvey and Liu \(2016\)](#) provide a bootstrap technique to model selection. Recently, [Freyberger et al. \(2017\)](#) propose a group LASSO procedure to select characteristics and to estimate how they affect expected returns nonparametrically. [Kozak et al. \(2017\)](#) use model-selection techniques to approximate the SDF and the mean-variance efficient portfolio as a function of all available test portfolios, and compare sparse models based on principal components of returns

with sparse models based on characteristics. They find that the cross-sectional explanatory power of a very sparse model based on few principal components (3-5) is higher than that of an equally-small model based on characteristics, and is instead comparable to a characteristics-based model with a larger number of factors (10-30). Consistent with these results, our empirical LASSO factor selection (that includes nontradable factors and tradable factors based on characteristics) tends to choose a few tens of factors (out of about 250).

The paper naturally builds on a large literature that has identified a variety of pricing factors, starting with the CAPM of [Sharpe \(1964\)](#) and [Lintner \(1965\)](#). Among the factors that the literature has proposed, some are based on economic theory (e.g., [Breedon \(1979\)](#), [Chen et al. \(1986\)](#), [Jagannathan and Wang \(1996\)](#), [Lettau and Ludvigson \(2001\)](#), [Yogo \(2006\)](#), [Pástor and Stambaugh \(2003a\)](#), [Adrian et al. \(2014\)](#), [He et al. \(2016\)](#)); others have been constructed using firm characteristics, such as [Fama and French \(1993, 2015\)](#), [Carhart \(1997\)](#), and [Hou et al. \(2014\)](#). Excellent reviews of cross-sectional asset pricing include [Campbell \(2000\)](#), [Lewellen et al. \(2010\)](#), [Goyal \(2012\)](#), and [Nagel \(2013\)](#).

We also build upon the econometrics literature devoted to the estimation and testing of asset pricing models using two-pass regressions, dating back to [Jensen et al. \(1972\)](#) and [Fama and MacBeth \(1973\)](#). Over the years, the econometric methodologies have been refined and extended; see, for example, [Ferson and Harvey \(1991\)](#), [Shanken \(1992\)](#), [Jagannathan and Wang \(1996\)](#), [Welch \(2008\)](#), and [Lewellen et al. \(2010\)](#). These papers, along with the majority of the literature, rely on large T and fixed n asymptotic analysis for statistical inference and only deal with models in which all factors are specified and observable. [Bai and Zhou \(2015\)](#) and [Gagliardini et al. \(2016\)](#) extend the inferential theory to the large n and large T setting, which delivers better small-sample performance when n is large relative to T . [Gagliardini et al. \(2017\)](#) further propose a diagnostic criterion to detect potentially omitted factors from the residuals of an observable factor model. [Connor et al. \(2012\)](#) use semiparametric methods to model time variation in the risk exposures as a function of observable characteristics, again when both n and T are large. [Giglio and Xiu \(2016\)](#) rely on a similar large n and large T analysis, but estimate risk premia (not risk prices as in this paper) in the case in which not all relevant pricing factors are observed. [Raponi et al. \(2016\)](#), on the other hand, study the ex-post risk premia using large n and fixed T asymptotics. For a review of this literature, see [Shanken \(1996\)](#), [Jagannathan et al. \(2010\)](#), and more recently, [Kan and Robotti \(2012\)](#).

A crucial distinction between our paper and the aforementioned literature is that we focus on the evaluation of a *new factor*, which may be motivated by economic theory, rather than testing or estimating an entire reduced-form asset pricing model. In fact, we point out that model selection mistakes are inevitable in any finite sample, so that making inference about the role of any factor just by looking at whether that factor is selected by LASSO or other model-selection techniques

will lead to erroneous inference. Instead, we do not take a stand on which model is correct: we use the model selection methods to approximate a part of the stochastic discount factor that serves as control, and we explicitly account for potential model selection mistakes when conducting inference. To the extent that the procedure is used to test a new factor g_t that is determined ex-ante and motivated by theory, it is not directly subject to the multiple testing concern that [Harvey and Liu \(2016\)](#) aim to address.¹ Our procedure also helps alleviate the concern of data-snooping, another form of multiple testing (see e.g., [Lo and MacKinlay \(1990\)](#), [Harvey et al. \(2015\)](#)), because we suggest imposing discipline to the selection of controls as opposed to the conventional practice of selecting arbitrary controls that leaves the researcher much more freedom.

Of course, the existing literature has routinely attempted to evaluate the contribution of new factors relative to some benchmark model, typically by estimating and testing the alpha of a regression of the new factor onto existing factors (e.g., [Barillas and Shanken \(2015\)](#) and [Fama and French \(2016\)](#)). Our methodology differs from the existing procedures in several ways. First, we do not use as control an arbitrary set of factors from h_t (e.g., the three Fama-French factors), but rather we select from h_t the control model that best explains the cross section of returns. In addition, our procedure aims to minimize the potential omitted variable bias while enhancing statistical efficiency. Second, we not only test whether the factor of interest g_t is useful in explaining asset prices, but we also estimate its role in driving marginal utility (its coefficient in the stochastic discount factor, or risk price); this is important to be able to interpret the results in economic terms and relate them to the models that motivated the choice of g_t . Third, our procedure handles both traded and non-traded factors. Fourth, our procedure leverages information from the cross section of the test assets in addition to the times-series of the factors. Lastly, our inference is valid given a large dimensional set of controls and test assets in addition to an increasing span of time series.

A recent literature has focused on various pitfalls in estimating and testing linear factor models. For instance, ignoring model misspecification and identification failure leads to an overly positive assessment of the pricing performance of spurious ([Kleibergen \(2009\)](#)) or even useless factors ([Kan and Zhang \(1999a,b\)](#); [Jagannathan and Wang \(1998\)](#)), and biased risk-premia estimates of true factors in the model. Therefore, the use of inference methods that are robust to model misspecification is more reliable ([Shanken and Zhou \(2007\)](#); [Kan and Robotti \(2008\)](#); [Kleibergen \(2009\)](#); [Kan and Robotti \(2009\)](#); [Kan et al. \(2013\)](#); [Gospodinov et al. \(2013\)](#); [Kleibergen and Zhan \(2014\)](#); [Gospodinov et al. \(2014\)](#); [Bryzgalova \(2015\)](#); [Burnside \(2016\)](#)). Existing literature considers the inference of pseudo-true parameters in the presence of model misspecification, whereas we correct the model misspecification bias and make inference about the original parameters.

¹The two methodologies could potentially be combined to produce more conservative inference that also deals with the possibility that the set of test factors g_t is selected ex-post after looking at the inference results, raising concerns about multiple testing. We leave this for future research.

Last but not least, our paper is related to a large statistical and machine-learning literature on variable selection and regularization using LASSO and post-selection inference. For theoretical properties of LASSO, see [Bickel et al. \(2009\)](#), [Meinshausen and Yu \(2009\)](#), [Tibshirani \(2011\)](#), [Wainwright \(2009\)](#), [Zhang and Huang \(2008\)](#), [Belloni and Chernozhukov \(2013\)](#). For the post-selection-inference method, see, for example, [Belloni et al. \(2012\)](#), [Belloni et al. \(2014b\)](#), and review articles by [Belloni et al. \(2014a\)](#) and [Chernozhukov et al. \(2015\)](#). Our asymptotic results are new to the existing literature in two important respects. First, our setting is a large panel regression with a large number of factors (p), in which both cross-sectional and time-series dimensions (n and T) increase. Second, our procedure in fact selects covariances between factors and returns, which are contaminated by estimation errors, rather than factors themselves that are immediately observable.

The rest of the paper is organized as follows. In [Section 2](#), we set up the model, present our methodology, and develop relevant statistical inference. [Section 3](#) provides Monte Carlo simulations that demonstrate the finite-sample performance of our estimator. In [Section 4](#), we show several empirical applications of the procedure. [Section 5](#) concludes. The appendix contains technical details.

2 Methodology

2.1 Model Setup

We start from a linear specification for the stochastic discount factor (SDF):

$$m_t := \gamma_0^{-1} - \gamma_0^{-1} \lambda_b^\top v_t := \gamma_0^{-1} (1 - \lambda_g^\top g_t - \lambda_h^\top h_t), \quad (1)$$

where γ_0 is the zero-beta rate, g_t is a $d \times 1$ vector of factors to be tested, and h_t is a $p \times 1$ vector of potentially confounding factors. Both g_t and h_t are de-measured; that is, they are factor innovations satisfying $E(g_t) = E(h_t) = 0$. λ_g and λ_h are $d \times 1$ and $p \times 1$ vectors of parameters, respectively. We refer to λ_g and λ_h as the *risk prices* of the factors g_t and h_t .

Our goal here is to make inference on the risk prices of a small set of factors g_t while accounting for the explanatory power of a large number of existing factors, collected in h_t . These factors are not necessarily all useful factors: their corresponding risk prices may be equal to zero. This framework potentially includes completely useless factors (factors that have a risk price of zero and whose covariances with returns are uncorrelated with the covariances of returns with the SDF), as well as redundant factors (factors that have a risk price of zero but whose covariances with returns are correlated in the cross section with the covariance between returns and the SDF).

We want to estimate and test the risk price of g_t for two reasons. First, it directly reveals whether g_t drives the SDF after controlling for h_t , that is, whether g_t contains additional pricing

information relative to h_t (Cochrane (2009)), or whether it is instead redundant or useless. Second, the coefficient λ_g indicates *how* g_t affects marginal utility conditional on other factors in the model. For example, a positive sign for λ_g tells us that states where g_t is low are high-marginal-utility states. The estimate of λ_g can therefore be used to test predictions of asset pricing models about how investors perceive g_t shocks.

In addition to g_t and h_t , we observe a $n \times 1$ vector of test asset returns, r_t . Because of (1), expected returns satisfy:

$$E(r_t) = \iota_n \gamma_0 + C_v \lambda_v = \iota_n \gamma_0 + C_g \lambda_g + C_h \lambda_h, \quad (2)$$

where ι_n is a $n \times 1$ vector of 1s, $C_a = \text{Cov}(r_t, a_t)$, for $a = g, h$, or v . Furthermore, we assume the dynamics of r_t follow a standard linear factor model:

$$r_t = E(r_t) + \beta_g g_t + \beta_h h_t + u_t, \quad (3)$$

where β_g and β_h are $n \times d$ and $n \times p$ factor-loading matrices, u_t is a $n \times 1$ vector of idiosyncratic components with $E(u_t) = 0$ and $\text{Cov}(u_t, v_t) = 0$.

Equation (2) represents expected returns in terms of (univariate) covariances with the factors, multiplied by risk prices λ_g and λ_h . An equivalent representation of expected returns can be obtained in terms of multivariate betas:

$$E(r_t) = \iota_n \gamma_0 + \beta_g \gamma_g + \beta_h \gamma_h, \quad (4)$$

where β_g and β_h are the factor exposures (i.e., multivariate betas) and γ_g and γ_h are the *risk premia* of the factors. Risk prices λ and risk premia γ are directly related through the covariance matrix of the factors, but they differ substantially in their interpretation. In this paper, we aim to estimate the *risk prices* of the factors g_t , not their risk premia. The risk premium γ of a factor tells us whether investors are willing to pay to hedge a certain risk factor, but it does not tell us whether that factor is useful in pricing the cross section of returns. For example, a factor could command a nonzero risk premium without even appearing in the SDF, simply because it is correlated with the true factors. As discussed extensively in Cochrane (2009), to understand whether a factor is useful in pricing the cross section of assets, we should look at its risk price λ , not its risk premium γ .

Because the link between risk prices and risk premia depends on the covariances among factors, it is useful to write explicitly the projection of g_t on h_t as

$$g_t = \eta h_t + z_t, \quad \text{where} \quad \text{Cov}(z_t, h_t) = 0. \quad (5)$$

Finally, for the estimation of λ_g , it is essential to characterize the cross-sectional dependence between C_g and C_h , so we write the cross-sectional projection of C_g onto C_h as:

$$C_g = \iota_n \xi^\top + C_h \chi^\top + C_e, \quad (6)$$

where ξ is a $d \times 1$ vector, χ is a $d \times p$ matrix, and C_e is a $n \times d$ matrix of cross-sectional regression residuals.²

2.2 Challenges with Standard Two-Pass Methods

Using two-pass regressions to estimate empirical asset pricing models dates back to [Jensen et al. \(1972\)](#) and [Fama and MacBeth \(1973\)](#). Partly because of its simplicity, this approach is widely used in practice. The procedure involves two steps, including one asset-by-asset time-series regression to estimate individual factor loadings β_s , and one cross-sectional regression of expected returns on the estimated factor loadings to estimate risk *premia* γ .

Because our parameter of interest is the risk price of g_t , λ_g , instead of the risk premium, the first step needs to be modified to use covariances between returns and factors rather than multivariate betas. In a low-dimensional setting, this method would work smoothly for the estimation of λ_g , as pointed out by [Cochrane \(2009\)](#).

However, the empirical asset pricing literature has created hundreds of factors, which can include useless and redundant factors in addition to useful factors; all and only the useful ones should be used as controls in estimating the risk price of newly proposed factors g_t and testing for their contribution to asset pricing (λ_g). Over time, the number of potential factors p discovered in the literature has increased to the same scale as, if not greater than, n or T . In such a scenario, the standard cross-sectional regression with all factor covariances included is at best highly inefficient, because the convergence rate in this regression is $p^{1/2}n^{-1/2}$. Moreover, if p is smaller than n yet of the same scale, asymptotic inference fails entirely to converge. When p is larger than n , the standard Fama-MacBeth approach becomes infeasible because the number of parameters exceeds the sample size.

Standard methodologies therefore do not work well if at all in a high-dimensional setting due to the curse of dimensionality, so that dimension-reduction and regularization techniques are inevitable for valid inference. The existing literature has so far employed ad hoc solutions to this dimensionality problem. In particular, in testing for the contribution of a new factor, it is common to cherry-pick a handful of control factors, such as the prominent Fama-French three factors, effectively imposing an assumption that the selected model is the true one (and is not missing any additional factors). However, this assumption is clearly unrealistic. These standard models have generally poor performance in explaining a large available cross section of expected returns beyond 25 size- and value-sorted portfolios, indicating omitted factors are likely to be present in the data. The stake of selecting an

²For the sake of clarity and simplicity, we assume the set of testing assets used is not sampled randomly but deterministically, so that these covariances and loadings are treated as non-random. This is without loss of generality, because their sampling variation does not affect the first-order asymptotic inference. By contrast, [Gagliardini et al. \(2016\)](#) consider random loadings as a result of a random sampling scheme from a continuum of assets.

incorrect model is high, because it leads to model misspecification and omitted variable bias when useful factors are not included, or an efficiency loss when useless or redundant factors are included.

2.3 Sparsity and LASSO

This issue is not unique to asset pricing. To address it, we need to impose a certain low-dimensional structure in the model. In this paper, we impose a sparsity assumption that has a natural economic interpretation and has recently been studied at length in the machine-learning literature. Imposing sparsity in our setting means that a relatively small number of factors exist in h_t , whose linear combinations along with g_t yield the SDF m_t , and those alone are relevant for the estimation of λ_g . More specifically, sparsity in our setting means there are only s non-zero entries in λ_h , and in each row of η and χ , where s is small relative to n and T . The sparsity assumption allows us to extract the most influential factors, while making valid inference on the parameters of interest, without *prior knowledge* or even *perfect recovery* of the useful factors that determine m_t .

Does sparsity make sense in asset pricing? In fact, the asset pricing literature has adopted the concept of sparsity without always explicitly acknowledging it. In addition to the proposed factor or the factor of interest, almost all empirical asset pricing models include only a handful of control factors, such as the Fama-French three or five factors, the momentum factor, etc. Such models provide a parsimonious representation of the cross section of expected returns, hence they typically outperform models with many factors in out-of-sample settings. This is a form of sparsity where the few factors allowed to have a non-zero risk price are chosen *ex ante*. Moreover, sparse models are easier to interpret and to link to economic theories, compared to alternative latent factor models, which often use the principal components as factors. Last but not least, as advocated in [Friedman et al. \(2009\)](#), one should “bet on sparsity” since no procedure does well in dense problems. The notion of sparse versus dense is relative to the sample size, the number of covariates, the signal to noise ratio, etc. Sparsity does not necessarily mean that the true model should always only involve a very small number of factors, say 3 or 5. More non-zero coefficients can be identified given better conditions (e.g., larger sample size), and in our empirical work we in fact find that twenty or more factors may be needed to achieve a good approximation of the SDF (still, a large reduction compared to the 300 potential factors in h_t).

To leverage sparsity, [Tibshirani \(1996\)](#) proposes the so-called LASSO estimator, which incorporates into the least-squares optimization a penalty function on the \mathbb{L}_1 norm of parameters, which leads to an estimator that has many zero coefficients in the parameter vector. The LASSO estimator has appealing properties in particular for prediction purposes. With respect to parameter estimation, however, a well-documented finite-sample bias is associated with the non-zero coefficients of the LASSO estimate because of the regularization. For these reasons, [Belloni and Chernozhukov \(2013\)](#)

and Belloni et al. (2012) suggest the use of a “Post-LASSO” estimator, which has more desirable statistical properties. The Post-LASSO estimator runs LASSO as a model selector, and then re-fits the least-squares problem without penalty, using only variables that have non-zero coefficients in the first step.

2.4 Single-Selection LASSO and Model Selection Mistakes

In the asset pricing context, the LASSO and Post-LASSO procedures could theoretically be used to select the factors in h_t with non-zero risk prices as controls for g_t , therefore accounting for the possibility that h_t contains useless or redundant factors. In fact, when the number of factors is large, LASSO and Post-LASSO will asymptotically recover the true model under certain assumptions.

Unfortunately, these procedures are not appropriate when we conduct *inference* about risk prices (e.g., about the price of g_t as in our context), because in any finite sample, we can never be sure LASSO or Post-LASSO will select the correct model from h_t , just like we cannot claim the estimated parameter values in a given finite sample are equal to their population counterparts. But if the model is misspecified, that is, if important factors from h_t are mistakenly excluded, inference about risk prices will be affected by an omitted variable bias. Therefore, standard LASSO or Post-LASSO regressions will generally yield erroneous inference about risk prices, as we confirm in simulations in Section 3.

This omitted variable bias due to model-selection mistakes is exacerbated if risk exposures to the omitted factors are highly correlated in the cross section with the exposures to g_t (even though these factors may have a small in-sample price of risk, which is why they may be omitted by LASSO). We will therefore need to ensure that these factors are included in the set of controls *even if LASSO would suggest excluding them*. Note this issue is not unique to high-dimensional problems – see, for example, Leeb and Pötscher (2005) – but it is arguably more severe in such a scenario because model selection is inevitable.

2.5 Two-Pass Regression with Double Selection LASSO

To guard against omitted variable biases due to selection mistakes, we therefore adopt a double-selection strategy in the same spirit as what Belloni et al. (2014b) propose for estimating the treatment effect. The first selection (basically, standard LASSO) searches for factors in h_t whose covariances with returns are useful for explaining the cross section of expected returns. A second selection is then added to search for factors in h_t potentially missed from the first step, but that, if omitted, would induce a large omitted variable bias. Factors excluded from both stages of the double-selection procedure must have small risk prices and have covariances that correlate only mildly in the cross section with the covariance between factors of interest g_t and returns – these factors can be excluded

with minimal omitted variable bias. This strategy results in a parsimonious model that minimizes the omitted factor bias ex ante when estimating and testing λ_g .

The regularized two-pass estimation proceeds as follows:

(1) Two-Pass Variable Selection

- (1.a) Run a cross-sectional LASSO regression of average returns on sample covariances between factors in h_t and returns:³

$$\min_{\gamma, \lambda} \left\{ n^{-1} \left\| \bar{r} - \iota_n \gamma - \widehat{C}_h \lambda \right\|^2 + \tau_0 n^{-1} \|\lambda\|_1 \right\}, \quad (7)$$

where $\widehat{C}_h = \widehat{\text{Cov}}(r_t, h_t) = T^{-1} \bar{R} \bar{H}^\top$.⁴ This step selects among the factors in h_t those that best explain the cross section of expected returns. Denote $\{\widehat{I}_1\}$ as the set of indices corresponding to the selected factors in this step.

- (1.b) For each factor j in g_t (with $j = 1, \dots, d$), run a cross-sectional LASSO regression of $\widehat{C}_{g, \cdot, j}$ (the covariance between returns and the j th factor of g_t) on \widehat{C}_h (the covariance between returns and all factors h_t):⁵

$$\min_{\xi_j, \chi_{j, \cdot}} \left\{ n^{-1} \left\| (\widehat{C}_{g, \cdot, j} - \iota_n \xi_j - \widehat{C}_h \chi_{j, \cdot}^\top) \right\|^2 + \tau_j n^{-1} \|\chi_{j, \cdot}^\top\|_1 \right\}. \quad (8)$$

This step identifies factors whose exposures are highly correlated to the exposures to g_t in the cross-section. This is the crucial second step in the double-selection algorithm, that identifies factors that may be missed by the first step but that may still induce large omitted variable bias in the estimation of λ_g if omitted, due to their covariance properties. Denote $\{\widehat{I}_{2, j}\}$ as the set of indices corresponding to the selected factors in the j th regression, and $\widehat{I}_2 = \bigcup_{j=1}^d \widehat{I}_{2, j}$.

(2) Post-selection Estimation

Run an OLS cross-sectional regression using covariances between the selected factors from *both* steps and returns:

$$(\widehat{\gamma}_0, \widehat{\lambda}_g, \widehat{\lambda}_h) = \arg \min_{\gamma_0, \lambda_g, \lambda_h} \left\{ \left\| \bar{r} - \iota_n \gamma_0 - \widehat{C}_g \lambda_g - \widehat{C}_h \lambda_h \right\|^2 : \lambda_{h, j} = 0, \quad \forall j \notin \widehat{I} = \widehat{I}_1 \cup \widehat{I}_2 \right\}. \quad (9)$$

We refer to this procedure as a double-selection approach, as opposed to the single-selection approach which only involves (1.a) and (2).

³We use $\|A\|$ and $\|A\|_1$ to denote the operator norm and the \mathbb{L}_1 norm of a matrix $A = (a_{ij})$, that is, $\sqrt{\lambda_{\max}(A^\top A)}$, $\max_j \sum_i |a_{ij}|$, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix.

⁴For any matrix $A = (a_1 : a_2 : \dots : a_T)$, we write $\bar{a} = T^{-1} \sum_{t=1}^T a_t$, $\bar{A} = A - \iota_n \bar{a}$.

⁵For any matrix A , we use $A_{i, \cdot}$ and $A_{\cdot, j}$ to denote the i th row and j th column of A , respectively.

The LASSO estimators involve only convex optimizations, so that the implementation is quite fast. Statistical software such as R and Matlab have existing packages that implement LASSO using efficient algorithms. Note that other variable-selection procedures are also applicable. Either (1.a) or (1.b) can be replaced by other machine-learning methods such as regression tree, random forest, boosting, and neural network, as shown in [Chernozhukov et al. \(2016\)](#) for treatment-effect estimation, or by subset selection, partial least squares, and PCA regressions (or with Lasso selection on top of PCs similar to [Kozak et al. \(2017\)](#)).

It is useful to relate our approach to the recent model selection method by [Harvey and Liu \(2016\)](#). Their model selection procedure is an algorithm that resembles the forward stepwise regression in [Friedman et al. \(2009\)](#) (a so-called “greedy” algorithm). Their algorithm evaluates the contribution of each factor relative to a pre-selected best model through model comparison, and builds up the best model sequentially. Just like LASSO cannot deliver the true model in a finite sample with certainty, this algorithm cannot do so either, because it makes commitments to certain variables too early which prevent the algorithm from finding the best overall solution later. For example, if one of the factors in the pre-selected model is redundant relative to the factor under consideration (i.e., the latter factor is in the DGP and the former one is a noisy version of it), the latter factor could either be added or discarded depending on how noisy the former factor is. Neither scenario, however, yields a model that is closer to the truth. Note that, in any case, if this algorithm were preferred to LASSO for any reasons, we could easily substitute it in place of LASSO and still obtain correct inference, because our procedure explicitly accounts for model selection mistakes.

Our LASSO regression contains nonnegative regularization parameters, for example, τ_j ($j = 0, 1, \dots, d$), to control the level of penalty. A higher τ_j indicates a greater penalty and hence results in a smaller model. The optimization becomes a least-squares problem if τ_j is 0. In practice, we typically test one factor each time, so that this procedure involves two regularization parameters τ_0 and τ_1 . To determine these parameters, we adopt the commonly used 5-fold cross-validation, BIC, and AIC; see, for example, [Friedman et al. \(2009\)](#). BIC tends to select a more parsimonious model than cross-validation and AIC.

We can also give different weights to λ_h . [Belloni et al. \(2012\)](#) recommend a data-driven method for choosing a penalty that allows for non-Gaussian and heteroskedastic disturbances. We adopt a strategy in the spirit of [Bryzgalova \(2015\)](#), which assigns weights to λ_h proportional to the inverse of the operator norm of the univariate betas of the corresponding factor in h_t . This strategy helps remove spurious factors in h_t because of a higher penalty assigned on those factors with smaller univariate betas.

2.6 Statistical Inference

We derive the asymptotic distribution of the estimator for λ_g under a jointly large n and T asymptotic design. Whereas d is fixed throughout, s and p can either be fixed or increasing. In the appendix, we prove the following theorem:

Theorem 1. *Under Assumptions A.1 - A.6 in Appendix A.2, if $s^2 T^{1/2}(n^{-1} + T^{-1}) \log(n \vee p \vee T) = o(1)$, we have*

$$T^{1/2}(\widehat{\lambda}_g - \lambda_g) \xrightarrow{\mathcal{L}} \mathcal{N}_d(0, \Pi),$$

where the asymptotic variance is given by

$$\Pi = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T \mathbb{E} \left((1 - \lambda^\top v_t)(1 - \lambda^\top v_s) \Sigma_z^{-1} z_t z_s^\top \Sigma_z^{-1} \right), \quad \Sigma_z = \text{Var}(z_t).$$

We stress this result holds even with imperfect model selection. That is, the selected models from (7) and (8) may omit certain useful factors and include redundant ones, which nonetheless has a negligible effect on the inference of λ_g . Using analysis similar to Belloni et al. (2014b), the results can be strengthened to hold uniformly over a sequence of data-generating processes that may vary with the sample size, so that our inference is valid without relying on perfect recovery of the correct model in finite sample. Moreover, the asymptotic distribution of $\widehat{\lambda}_g$ does not rely on covariances or factor loadings of g_t and h_t , because they appear in strictly higher-order terms, which further facilitates our inference. The next theorem provides a Newey-West-type estimator of the asymptotic variance Π .

Theorem 2. *Suppose the same assumptions as in Theorem 1 hold. In addition, Assumption A.7 holds. If $qs^{3/2}(T^{-1/2} + n^{-1/2}) \|V\|_{\text{MAX}} \|Z\|_{\text{MAX}} = o_p(1)$,⁶ we have*

$$\widehat{\Pi} \xrightarrow{p} \Pi,$$

where $\widehat{\lambda} = (\widehat{\lambda}_g : \widehat{\lambda}_h)$ is given by (9), and

$$\begin{aligned} \widehat{\Pi} &= \frac{1}{T} \sum_{t=1}^T (1 - \widehat{\lambda}^\top v_t)^2 \widehat{\Sigma}_z^{-1} \widehat{z}_t \widehat{z}_t^\top \widehat{\Sigma}_z^{-1} \\ &\quad + \frac{1}{T} \sum_{k=1}^q \sum_{t=k+1}^T \left(1 - \frac{k}{q+1} \right) \left((1 - \widehat{\lambda}^\top v_t)(1 - \widehat{\lambda}^\top v_{t-k}) \widehat{\Sigma}_z^{-1} (\widehat{z}_t \widehat{z}_{t-k}^\top + \widehat{z}_{t-k} \widehat{z}_t^\top) \widehat{\Sigma}_z^{-1} \right), \\ \widehat{\Sigma}_z &= \frac{1}{T} \sum_{t=1}^T \widehat{z}_t \widehat{z}_t^\top, \quad \widehat{z}_t = g_t - \widetilde{\eta}_{\widetilde{I}} h_t, \quad \widetilde{\eta}_{\widetilde{I}} = \arg \min_{\eta} \left\{ \|G - \eta H\|^2 : \eta_{\cdot j} = 0, \quad j \notin \widetilde{I} \right\}, \end{aligned}$$

⁶We use $\|A\|_{\text{MAX}}$ to denote the \mathbb{L}_∞ -norm of A in the vector space.

and \tilde{I} is the union of selected variables using a LASSO regression of each factor in g_t on h_t :

$$\min_{\eta_j} \left\{ T^{-1} \|G_{j\cdot} - \eta_j H\|^2 + \bar{\tau}_j T^{-1} \|\eta_j\|_1 \right\}, \quad j = 1, 2, \dots, d. \quad (10)$$

3 Simulation Evidence

One of the central advantages of our double-selection method is that it produces proper inference on the risk price λ_g of a factor, taking explicitly into account the possibility that the model-selection step (based on LASSO) may mistakenly include some irrelevant factors or exclude useful factors in any finite sample.

In this section, we therefore study the finite-sample performance of our inference procedure using Monte Carlo simulations. In particular, we show that if one were to make inference on λ_g by selecting the control factors via standard LASSO (and ignoring potential mistakes in model selection), the omitted variable bias resulting from selection mistakes would yield incorrect inference about λ_g . Instead, our double-selection procedure fully corrects for this problem in a finite sample and produces valid inference. In what follows, we first give details of the simulation procedure and then show the results of the Monte Carlo experiment.

3.1 Simulating the Data-Generating Process

We are interested in making inference on λ_g , the vector of prices of risk of three factors in g_t . g_t includes a useful factor (denoted as g_{1t}) as well as a useless factor and a redundant factor (denoted together as a 2×1 vector g_{2t}). g_{2t} has zero risk price, that is, $\lambda_{g_2} = 0$, but the covariance of the redundant factor is correlated with the cross section of expected returns. In our simulation, h_t is a large set of factors that includes 4 useful factors h_{1t} , and $p-4$ useless and redundant factors collected in h_{2t} (so the total dimension of h_t is p).

We give the step-by-step detail of DGP simulation in [Appendix B](#). The steps described therein simulate the true model, a 5-factor model based on the factors h_{1t} and g_{1t} . The simulated useless and redundant factors in g_{2t} and h_{2t} are correlated with the true factors: so they will command risk *premia* simply due to this correlation, even though they have zero risk prices because they do not affect marginal utility once the true factors are controlled for.

We calibrate our DGP to mimic the actual Fama-French 5-factor model. In particular, we calibrate χ , η , λ , Σ_z , the mean and covariance matrices of C_e , C_{h_1} , as well as Σ_{h_1} to match the summary statistics (time series and cross-sectional R^2 , factor-return covariances, etc.) of the Fama-French five factors estimated using characteristic-sorted portfolios, described in detail in [Section 4](#). We calibrate a diagonal Σ_u to match the average time series R^2 for this 5-factor model. For

redundant and useless factors, we calibrate their parameters using all the other factors in our data library, again described in detail in Section 4.

The total number of Monte Carlo trials is 5,000. Because we assume non-random selection of assets and that the randomness in the selection of test assets does not affect the inference to the first order, we simulate only once C_g , C_h , and hence β_g , β_h , so that they are constant throughout the rest of the Monte Carlo trials.

3.2 Simulation Results

We report here the results of various simulations from the model. We consider various settings with number of total factors $p = 100, 300$, number of assets $n = 100, 300, 500$, and length of time series $T = 360, 480, 600$. Also, we report simulation results using different choices of regularization parameters for robustness, including BIC, AIC, and 5-fold cross-validation.

Figure 1 compares the asymptotic distributions of the proposed double-selection estimator with that of the single-selection estimator for the case $p = 300$, $n = 500$, and $T = 600$. The right side of the figure shows the distribution of the t-test for the price of risk λ_g of the three factors (useful in the first row, redundant in the second row, and useless in the third row) when using the controls selected by standard LASSO (i.e., a single-selection-based estimator). The panels show that inference without double-selection adjustment displays substantial biases and distortion from normality. The left side of the figure shows instead that our double-selection procedure produces an unbiased and asymptotically normal test, as predicted by Theorem 1.

Figure 2 plots the frequency with which each of the simulated factors is selected across simulations (with each bar corresponding to a different simulated factor, identified by its ID from 1 to 300). The top panel corresponds to the factors selected in the first LASSO selection, the second panel corresponds to the factors selected in the second selection, and the last panel corresponds to the union of the two.

Note that by construction, the true factors in h_t are the first 4 (the fifth true factor is part of g_t). So if model selection were able to identify the right control factors in all samples perfectly, the first 4 bars should read 100%, while all other bars (corresponding to factors 5-300) should read 0%.

That is not the case in the simulations. While some factors are often selected by LASSO (top panel), not all are: factor 1 is only selected in about 50% of the samples, and factor 3 less than 40% of the samples. Therefore, in a large fraction of samples, the control model would be missing some true factors, generating the omitted variable bias displayed in Figure 1. At the same time, LASSO often includes erroneously spurious factors – often more than 20 factors in total. The key to correct inference that our procedure achieves is that the two-step selection procedure minimizes the

potential omitted factor bias.

Tables 1, 2, and 3 compare the biases and root-mean-squared errors (RMSEs) for double-selection, single-selection, and the OLS estimators of each entry of λ_g , respectively. We only report results based on BIC and 5-fold cross-validation for selecting regularization parameters. The results based on AIC is similar to those of the cross-validation.

Both double- and single-selection estimators outperform OLS in terms of the RMSEs, particularly when p is large relative to n . When p is greater than n , OLS becomes infeasible. This result confirms the efficiency benefits of dimension-reduction techniques. In addition, the double-selection estimator has a smaller bias and a smaller RMSE than the single-selection estimator. But the main advantage of double-selection relative to single-selection is in removing the distortions to inference, visible from the distribution of standardized statistics in Figure 1.

Note the biases and RMSEs become smaller as n and T increase. Instead, when p is larger, the results exacerbate slightly. Overall, the simulation results confirm our econometric analysis: the double-selection estimator outperforms the benchmarks.

4 Empirical Analysis

In this section we apply our methodology to our data library of hundreds of factors. We start by showing how our estimation procedure can be used to evaluate whether a newly proposed factor provides useful pricing information compared to the myriad of existing factors. We document that indeed some of the recently proposed factors (e.g., profitability and investment) contribute significantly to explaining asset prices, even controlling for a large number of existing factors in the literature. At the same time, many factors introduced in the last few years appear entirely redundant and contain no new useful information for pricing the cross section of returns.

Next, we compare our procedure to standard model selection methods for the purpose of understanding whether specific factors are useful. We show that models selected by LASSO (like other statistical selection methods) are not stable in finite samples: they vary greatly with the sample in which the estimation is performed and with the choice of tuning parameters. In other words, statistical model selection is prone to making mistakes in choosing factors in finite samples, and cannot be relied upon to make stable inference about the *identities* of the factors in a model. On the contrary, our testing procedure (that makes use of LASSO but explicitly accounts for model selection mistakes) produces inference that is substantially more robust.

We then perform a recursive evaluation exercise, in which we test the factors in the library as they were historically introduced in the literature, relative to the ones existing at the time; we show that had our estimator been used over time to evaluate the new factors over the last 20 years, most

of the factors in our data library would have been deemed useless or redundant right when they were introduced, thus bringing discipline to the zoo of factors.

4.1 Data

4.1.1 The Zoo of Factors

Our factor library contains 99 risk factors at the monthly frequency for the period from July 1980 to December 2016, obtained from multiple sources. First, we download all workhorse factors in the U.S. equity market from Ken French’s data library. Then we add several published factors directly from the authors’ websites, including liquidity from [Pástor and Stambaugh \(2003a\)](#), the q-factors from [Hou et al. \(2014\)](#),⁷ and the intermediary asset pricing factors from [He et al. \(2016\)](#). We also include factors from the AQR data library, such as Betting-Against-Beta, HML Devil, and Quality-Minus-Junk. In addition to these 17 publicly available factors, we follow [Fama and French \(1993\)](#) to construct value-weighted portfolios and 82 long-short factors using those firm characteristics surveyed in [Green et al. \(2016\)](#).⁸

More specifically, we include only stocks for companies listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. Moreover, we exclude financial firms and firms with negative book equity. For each characteristic, we sort stocks using NYSE breakpoints based on their previous year-end values, then build and rebalance a long-short value-weighted portfolio (top 30% - bottom 30% or 1-0 dummy difference) every June for a 12-month holding period. Both [Fama and French \(2008\)](#) and [Hou et al. \(2016\)](#) have discussed the importance of using NYSE breakpoints and value-weighted portfolios. Microcaps, i.e., stocks with market equity smaller than the 20th percentile, have the largest cross-sectional dispersion in most anomalies, while accounting for only 3% of the total market equity. Equal-weighted returns overweight microcaps, despite their small economic importance.

In Table [A.1](#) of the Appendix, we report a complete list of the 99 factors and various descriptive statistics (publication year, monthly average returns for tradable factors, and annualized Sharpe ratios), as well as the academic references. We follow [Hou et al. \(2014\)](#) and provide six main categories for the factor classification: Momentum, Value-versus-Growth, Investment, Profitability, Intangibles, and Trading Frictions. In our factor zoo, each category contains at least seven factors. Furthermore, to capture the potential nonlinearity of the SDF (and consistent with empirical evidence, e.g., [Freyberger et al. \(2017\)](#) and [Kozak et al. \(2017\)](#)) we add as controls 197 factors that include 99 squared terms for these primary risk factors, and 98 interaction terms between Small Minus Big and each other factor.

⁷We are grateful to Lu Zhang for sharing the updated factors data.

⁸We are grateful to Jeremiah Green for sharing the firm-characteristics SAS calculation code.

4.1.2 Test Portfolios

We conduct our empirical analysis on a large set of standard portfolios of U.S. equities. We target U.S. equities because of their better data quality and because they are available for a long period; however, our methodology could be applied to any set of countries or asset classes. We focus on portfolios rather than individual assets because characteristic-sorted portfolios have more stable betas, higher signal-to-noise ratios, and they are less prone to missing data issues, despite the existence of a bias-variance trade-off between the choice of portfolios and individual assets. Selecting a few portfolios based on sorts of a handful characteristics is likely to tilt the results in favor of these factors, see [Harvey and Liu \(2016\)](#), which is something we specifically address in our robustness tests. There might also be a loss in efficiency in using a few such portfolios, e.g., [Litzenberger and Ramaswamy \(1979\)](#). In line with the suggestion of [Lewellen et al. \(2010\)](#), we base our analysis on a large cross section of characteristic-sorted portfolios, which helps strike a balance between having many individual stocks or a handful of portfolios.

We use a total of 1,825 portfolios as test assets. We start from a standard set of 175 portfolios: 25 portfolios sorted by size and book-to-market ratio, 25 portfolios sorted by size and beta, 25 portfolios sorted by size and operating profitability, 25 portfolios sorted by size and investment, 25 portfolios sorted by size and short-term reversal on prior (1-1) return, 25 portfolios sorted by size and momentum on prior (2-12) return, and 25 portfolios sorted by size and long-term reversal on prior (13-60) return. This set of test assets – all available from Kenneth French’s website – captures a vast cross-section of anomalies and exposures to different factors.⁹

We add to these 175 portfolios 1,650 additional ones obtained from our factor zoo, that cover additional characteristics. In particular, we include sets of 5×5 bivariate-sorted portfolios from all continuous factors in our factor zoo. The sorting procedure is same as that for the construction of factors, except that the stock universe is divided into five groups for each characteristic. For each firm characteristic, the bivariate-sorted 5×5 portfolios are constructed by intersecting its five groups with those formed on size (market equity). Notice that, the number of stocks in each 5×5 group can be unbalanced in the bivariate intersection. We only include the resulting portfolios if each of the 25 groups contains a sufficient number of stocks (at least 5). This procedure gives us 66 sets of 25 bivariate-sorted portfolios, yielding 1,650 portfolios.¹⁰

⁹See the description of all portfolio construction on Kenneth French’s website: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

¹⁰There are 16 factors for which bivariate-sorted portfolios are not available. 9 of 16 are dummy or categorical characteristics, including Dividend Initiation (25), Dividend Omission (26), New Equity Issue (27), Financial Statements Score (42), Number of earnings increases (38), # Years Since First Compustat Coverage (57), Financial Statement Performance (60), Sin Stocks (78), and Convertible Debt Indicator (99). 2 of 16, including Dividend to Price (4), and R&D Increase (54) have more than one fifth of characteristic values being zero for many years. The remaining 5 of 16,

As a robustness check, we have also created multiple sets of sequential-sorted portfolios using size and the other characteristics. The portfolios, which are also constructed at the end of each June, are created by first allocating stocks to five size groups. Stocks in each size group are then assigned to five sub-groups for the other characteristics using quantile breakpoints specific to that size group. This sorting gives 70 sets of 5×5 sequential-sorted portfolios, because we now have portfolios associated with 5 characteristics (8, 43, 44, 48, and 41) that have large correlations with size, but lose portfolios for Industry Momentum (37) (one of its 25 portfolio groups has less than 5 stocks.). A main advantage for the sequential-sorted portfolios is the numbers of stocks are more balanced in each 5×5 group than the bivariate-sorted portfolios.

4.2 Are New Factors Useful?

In this section we apply our methodology to factors that have been proposed in the last five years (since 2011), drawing the “control” factors from the set of more than 80 factors that were proposed before 2011, as well as the squares of these factors and interactions between the factors and size, for a total of 248 control factors (note that squares and interactions are nontradable factors).¹¹ We therefore ask whether the recently introduced factors add any new pricing information to the existing tradable and nontradable factors, or are redundant or outright useless in pricing the panel of returns. We have no ex-ante reason to expect the results to go in either direction. On the one hand, given that the set of potential control factors is already extremely large, one might think that new factors are unlikely to contribute much to pricing the cross section of returns. On the other hand, we expect new research to potentially uncover better factors over time, yielding factors that improve over the existing ones.

Table 4 reports the results for the factors proposed in the last five years, among which we find Quality-Minus-Junk (QMJ), Betting-Against-Beta (BAB), two investment factors (CMA from Fama-French and IA from HXZ), two profitability factors (RMW from Fama-French and ROE from HXZ), the nontradable intermediary capital factor from He et al. (2016), and several factors constructed on accounting measures.

For each factor, we estimate its price of risk and test its significance. The estimated risk price is directly informative about how that factor enters the stochastic discount factor, that is, how investors’ marginal utility depends on that factor. This has implications for asset pricing theories, that typically predict whether a factor increases or decreases investor’s marginal utility. At the same

including Industry-Adjusted Size (41), Bid-Ask Spread (8), Dollar Trading Volume (43), Volatility for Dollar Trading Volume (44), and Illiquidity (48), have either large positive (41) or large negative correlations (8, 43, 44, 48) with size (e.g., small firms are rarely liquid stocks), so that part of their 25 portfolios are missing.

¹¹More precisely, we have 83 factors introduced up to 2010, 83 squared terms, and 82 interactions with size (size is one of the 83 factors).

time, a non-zero price of risk indicates that the factor is useful in explaining the cross-section of expected returns (the equivalence is discussed in detail in [Cochrane \(2009\)](#)).

The table contains five columns of results, each reporting the point estimate of the risk price and the corresponding t-statistic. More specifically, the point estimate corresponds to the estimated slope of the cross-sectional regression of returns on (univariate) betas for each factor, using different methodologies to select the control factors: it represents the estimated average excess return in basis points per month of a portfolio with unit *univariate* beta with respect to that factor. This number is equal to the risk price λ_g but scaled to correspond to a unit beta exposure for ease of interpretation. A positive estimate for the risk price indicates that high values of the factor capture states of low marginal utility (good states of the world). The t-statistic in each column corresponds to the test of the hypothesis that the slope is equal to zero, constructed using different methodologies across columns.

The first column reports our main result – the estimates of risk prices for the factors introduced since 2011, with corresponding t-statistics, obtained with our double-selection (DS) procedure.¹² Most of the new factors appear statistically insignificant – our test therefore deems them redundant or useless relative to the factors introduced up to 2011. However, we still find a few important factors useful in explaining the cross-section, as their estimated risk price is significantly different from zero: in particular, QMJ, profitability (both the version of HXZ and that of Fama and French), and investment (both the HXZ and Fama-French versions). The estimated risk prices indicate that states of high marginal utility correspond to low values for all these factors. These results show that our double-selection method can discriminate between useful and redundant factors even when the set of controls contains hundreds of factors.

The second set of results reports the estimates that one would obtain using the naive single-selection (SS) methodology – that is, simply using LASSO to select the factors to use as controls, without the second selection step that is useful to avoid the omitted variable bias due to mistakes in model selection. The results are quite different from the double-selection approach, with different factors (maximum return and organizational capital) appearing significant; none of the factors that appear significant with the DS method do so when using SS. Given our discussion in the previous sections, it should not be surprising that results obtained using the SS method differ from those obtained using the DS method: our theoretical results and simulations show that the SS method is biased in finite samples. This table shows that these biases play a major role empirically.

The third column shows instead what the risk price estimates for the various factors would be if one simply used the Fama-French 3 factors (Market, SMB, HML) as controls, rather than selecting the controls optimally among the myriad of potential factors. The results differ noticeably from the

¹²Here and in the rest of the analysis we choose tuning parameters using 5-fold cross-validation.

benchmark with double selection. Of course, if the *true* SDF was known ex ante, selecting all and only the true factors as controls would lead to the most efficient estimate for the price of risk of g_t . In practice, however, it is unlikely that we can pin down the entire SDF with certainty. The aim of our double-selection procedure is precisely to select the controls statistically – avoiding arbitrary choices of control factors – while at the same time minimizing the potential omitted variable bias.

The fourth column shows one more alternative way to compute risk prices: using standard OLS estimation including in the cross-sectional regression *all* the hundreds of potential controls. This panel therefore shows what happens if no selection is applied at all on the factors. As discussed in the previous sections, this approach is unbiased but inefficient. We expect therefore (and confirm in the table) that the results appear much more noisy and the estimates less significant than when operating dimension reduction through our DS method. This result highlights the importance of dimension reduction methods when sorting through the myriad of existing factors.

The last column of the table shows the average excess return of the tradable factors, that is, their risk premium. This number represents the compensation investors obtain from bearing exposure to that factor, holding exposures to all other risk factors constant. As discussed, for example, in [Cochrane \(2009\)](#), the risk premium of a factor does *not* correspond to its ability to price other assets, that is, its coefficient in the SDF. Using the risk premium to assess the importance of a factor in a pricing model would be misleading. For example, consider two factors that are both equally exposed to the same underlying risk, plus some noise. Both factors will command an identical risk premium. Yet those factors are not *both* useful to price other assets—regardless of their level of statistical significance. The most promising way to reduce the proliferation of factors is not to look at their risk premium (no matter how significant it is), but to evaluate whether they add any pricing information to the existing factors. Our paper proposes a way to make this feasible even in a context of high dimensionality, when the set of potential control factors is large. We come back to this point in [Section 4.5](#).¹³

To sum up, [Table 4](#) shows that which factors are chosen as controls, and which econometric procedure is used for estimation, make a large difference for the conclusions about the risk price and the usefulness of factors. Both the theoretical analysis and the simulations provided in this paper suggest that the DS method allows researchers to make full use of the information in the existing zoo of factors without introducing biases while accounting for efficiency losses.

¹³It is interesting to note that about half of these factors do not have a significant risk premium, while they typically did in the original publications. This is partly due to the different sample period used here, and partly because we use a unified sorting methodology in this paper, rather than the heterogeneous methods used in the original papers. This result is consistent with the findings of [Hou et al. \(2016\)](#).

4.3 Economic Interpretation and Model Selection

The core idea of this paper is that we can make inference about the risk price and importance of a specific factor g_t – motivated by economic theory – even if we do not know exactly what the other true factors are, and even if we can never be sure to have recovered the correct model via statistical model selection methods. Therefore, the exact identity of the factors selected as controls by LASSO is not of primary importance for our analysis. For completeness, we report here the list of the 14 factors (out of 248) selected from h_t as controls: Excess Market Return, Sales to Inventory, Sales to Receivables, High Minus Low, Short-Term Reversal, Momentum, Industry-Adjusted Book to Market, Growth in Long Term Net Operating Assets, Change in 6-Month Momentum, Change in Capital Expenditures, Return on Invested Capital, Accrual Volatility, (Cash Flow to Debt)², (Change in Shares Outstanding)².

To the 14 factors selected in the first stage and reported, our double-selection procedure adds additional control factors in the second stage; these additional controls are those whose risk exposures are cross-sectionally correlated with those of the target factor g_t , and are crucial to minimize the omitted variable bias in risk prices, as demonstrated in the theory and simulation sections. Due to space constraints, we do not report the additional factors for all the g_t of Table 4: each factor g_t induces a different second-stage selection (whereas the first selection of controls, the set reported above, is common to all target factors).

It is important to remark that the list of factors selected from h_t by LASSO does not have (nor does it need to have) a direct economic interpretation. The objective of machine learning and model selection techniques is to reduce the dimensionality and maximize explanatory power of the low-dimensional model – not to provide an economically motivated selection of factors.

In many economic applications, the inability to interpret economically the selected model is an important shortcoming. In this paper, however, we only use model selection techniques to select the *controls* for the cross-sectional regression: that is, to approximate that part of the SDF that is unknown to the researcher, while recovering the true coefficient of the SDF on g_t . The factor of interest g_t therefore retains its economic interpretation (under the theory from which it was derived), and so do the sign and magnitude of its estimated risk price: the estimated coefficient λ_g directly tells us how that factor affects investors’ marginal utility, holding all the other factors in the SDF constant. We thus aim to combine the strength of machine learning methods with the interpretability obtained from economic theory that motivates the choice of the target factors g_t .

4.4 Stable Inference v.s. Unstable Model Selection

Statistical model selection methods not only make it hard to give an economic interpretation to the selected models; they are also prone to mistakes in selecting the factors, and the resulting models can be unstable with respect to the particular sample used and choice of tuning parameters. In this section, we show empirically that instead – as predicted by the theory of Section 2 – our double-selection inference is remarkably stable, and it is so despite the instability of the statistical model selection steps used to choose the controls from h_t .

We start by exploring how the model selected by LASSO (from h_t) and the inference about target factors g_t using our double-selection method depend on the tuning parameters. Recall that each dimension-reduction step via LASSO depends on one tuning parameter. Our double-selection procedure uses LASSO in two separate steps, so two tuning parameters are needed. To produce our benchmark estimates in Table 4, we choose tuning parameters using a cross-validation criterion. Here we show that the inference about λ_g is robust to changes in both tuning parameters, even as the set of factors selected from h_t by LASSO varies dramatically.

For each of the factors g_t introduced since 2011, we compute its double-selection t-statistic and the number of factors selected by the two LASSO steps from h_t , for a wide range of values of both tuning parameters. We then report them in Figure 3 using heatmaps. In the figure, each row corresponds to a different factor g_t . The left panel reports t-statistics, and the right panel reports the number of control factors selected from h_t . The two axes correspond to values for the two tuning parameters (in logs).

From the right panel, it is evident that our range of tuning parameters corresponds to a wide range of possible model dimensions. Consider for example the case of HXZ Profitability. Depending on the tuning parameter, the control model has anywhere between 10 and 60 factors: the factors selected by LASSO therefore vary considerably as a function of the tuning parameters; this is one way to see the instability of statistical model selection.

The left panels, however, show very high consistency in the inference about the economically-motivated factors g_t , for the entire range of tuning parameters. For example, for the HXZ Profitability factor, the t-statistic is always in the range of 2 to 4, whereas for insignificant factors like employee growth, the t-statistics are below significance levels across the entire range.

Overall, Figure 3 shows that the DS procedure makes inference robust to changes in the tuning parameters, even when the LASSO-selected model is instead quite sensitive to these changes.

Next, we explore the difference between the DS inference and LASSO across subsamples of our data. In this exercise, we bootstrap our data over different time periods and test assets (with

replacement on both dimensions), and perform our analysis on each bootstrap sample. For each sample, we select the control model using LASSO (re-estimating the tuning parameters in that sample only), then add factors from the second selection, and finally estimate the risk price of each target factor in that sample using our DS procedure. We construct in this way 2,000 bootstrap samples.

Figure 4 shows the distribution of t-statistics using our double-selection method for the factors introduced since 2011, across the 2,000 bootstrap samples. Recall that most of the factors in Table 4 are not significant: for those factors, we would then expect the t-statistic to be centered around 0 and the distribution of the t-statistic to lie mostly in the $(-2, +2)$ interval. Instead, we would expect factors that appear significant in our DS test to have a consistently positive (or negative) t-statistic: if the estimate of the risk price is stable across different samples, we would expect the distribution to lie mostly on one side of 0 (the side of course would depend on whether the risk price is positive or negative).

The figure shows that this is indeed the case. For the factors that do not appear significant in Table 4, the distribution of t-statistics across samples is centered around zero. For the factors that appear to be significant in Table 4 (for example, investment, profitability and QMJ) the distribution is well shifted to the right.

These results indicate that our risk price estimator gives very consistent results across samples. This is true even when the first model-selection step (based on standard LASSO) instead displays high variability across subsamples: Figure 5 shows, for each factor identified by its ID, in how many of the 2,000 samples the factor is actually selected as part of the model by simple LASSO. The figure shows striking variability in the model selection step. None of the 248 factors, except for the Market (No. 1) and Sales to Inventory (No. 16) are actually selected in more than 70% of the samples, and most of the factors are selected in 5% to 30% of the subsamples, but not in the others.

If LASSO had been able to perfectly select the true model, we should have found a small number of factors (say, 5 or 10) to be selected 100% of the time, and the remaining factors to be selected 0% of the time. Instead, LASSO clearly has difficulty pinning down *which* factors are the correct ones, selecting sometimes a group of them, other times a different group.

This exercise should caution us against using LASSO to decide whether a factor should be included in the SDF or not. A naive approach to deciding whether a new factor is useful in explaining the cross-section of asset prices above and beyond the existing factors would be to simply run a LASSO model selection procedure on the set of all factors (old and new), and check whether the new factor gets selected. But this approach does not lead to proper inference: the LASSO estimates will make mistakes in selecting the model, and this simple rule of thumb does not appropriately take the

possibility of model selection mistakes into account. On the contrary, our double-selection procedure corrects for these mistakes, producing an unbiased and efficient estimator of λ_g and correct inference.

4.5 Evaluating Factors Recursively: Risk Premia, Risk Prices, and t-Statistics thresholds

One of the motivations for using our methodology is that it can help distinguish useful from useless and redundant factors as they are introduced in the literature. Over time, this should help limit the proliferation of factors, and retain only those new factors that actually contain novel information to price the cross-section.

To illustrate this point, in each year starting in 1994 we consider the factors introduced during that year, and use our double-selection procedure to test whether they are useful or redundant relative to factors existing up to then. Note that the exercise is fully recursive, using only information available up to time t when evaluating a factor introduced at time t , both in choosing the set of potential controls h_t and in constructing the test portfolios (which are therefore sorted on characteristics introduced in the literature up to time t).

Table 5 reports the results. In the table, the factors introduced since 1994 are identified by their ID; the table underlines the ones that appear to be statistically significant according to our test, relative to the factors introduced before them. The table also reports the number of test assets used in each year and the number of control factors (that include squares and interactions) in h_t .

The results show that had our DS test been applied year by year starting in 1994, only 14 factors would have been considered useful, and a large majority would have been identified as redundant or useless.

It is useful to think about this exercise in light of the recent literature (e.g., [McLean and Pontiff \(2016\)](#), [Harvey et al. \(2015\)](#)) that has highlighted and tried to address the existence of a multitude of seemingly significant anomalies. The literature has proposed a variety of approaches, including adopting a stricter requirement for significance (such as using a threshold for the t-statistic of 3). Although the overarching theme is to tame the factor zoo, the perspectives are slightly different. The aforementioned papers emphasize the bias of data-snooping or raise the concern of multiple testing, whereas our focus is on omitted controls. All these problems could contribute to the proliferation of factors.

Our approach differs from the proposals in the existing literature in four substantial ways. First, and most important, we explicitly address the problem of omitted variable bias due to potential model selection mistakes when making inference about factors' contribution to asset prices. Second, our method directly takes into account the correlation among factors, rather than considering factors

individually and using Bonferroni-type bounds to assess their joint significance. We provide a statistical test of a factor’s contribution with desirable asymptotic properties, as demonstrated in the previous sections, and do not rely on simulation or bootstrap methods whose statistical properties in this context are unknown. Third, our method is specifically designed to handle hundreds of factors as controls, exploiting model-selection econometric advances to reduce the dimensionality of the factor set. Fourth, the criterion we employ for selecting factors is based on the risk price, not the risk premium of the factors (see a more detailed discussion on their differences in Section 4.2), as it is the right quantity to evaluate the contribution of a factor to explaining asset prices.

The various approaches that have been proposed in the literature so far address complementary issues to be overcome on the path to disciplining the zoo of factor. We leave for future research refinements of these methods that can potentially combine insights from our work and other recent papers.

4.6 Robustness Checks

In this section we further explore the robustness of our results, with respect to the test assets used for the estimation, the set of potential control factors h_t , and the dimension-reduction methodology used to select the control factors. We focus our robustness tests on the evaluation of recent factors (Table 4).

Column (1) of Table 6 reports our baseline results for convenience (as in the first column of Table 4). In column (2) we show results that use a sequential-sorting procedure to construct the 5×5 characteristic-sorted portfolios so as to demonstrate robustness to missing data. The results are consistent with our baseline specification, except that one more factor becomes significant.

Next, in column (3) we consider using only test portfolios constructed from characteristics available before 1994 when estimating the risk prices of all factors. The goal of this test is to highlight the fact that our results do not depend on including, among the test assets, portfolios sorted on the same characteristics on which the factors themselves were built. While the number of available test portfolios is smaller in this case, i.e., 450, the factors that appear significant are the same as in the baseline case.

We then conduct a robustness test on the set of potential control factors h_t . In column (4), we repeat our analysis including only tradable factors in the set of controls (that is, excluding factors that are nontradable as well as interactions and squared factors). Again, the results are consistent with our benchmark.

Finally, the last column shows that our results hold also when using a different dimension-reduction procedure. Which method is preferred in each context depends on the underlying model

assumptions, and given the assumptions we make, LASSO would be the most suitable model-selection method. However, Elastic Net is a reasonable alternative to explore in this context: it combines a penalty from LASSO with that of the Ridge regression. The model selected by the Elastic Net is naturally larger, but, as column (5) in the table shows, the results are consistent with our benchmark based on pure LASSO.

Overall, while across the different robustness tests the significance of some factors changes slightly, the main conclusions of Table 4 appear quite robust to these changes in specification. The table confirms that several of the factors introduced recently (like investment, profitability, and QMJ) have significant additional pricing power relative to all factors introduced in the literature before 2011.

5 Conclusion

In this paper we propose a regularized two-pass cross-sectional regression approach to establish the contribution to asset pricing of a factor g_t relative to a set of control factors h_t , where the potential control set can have high dimensionality and include useless or redundant factors. Our procedure uses recent model-selection econometric techniques (specifically the double-selection procedure of Belloni et al. (2014b)) to systematically select the best control model out of the large set of factors, while explicitly taking into account that in any finite sample we cannot be sure to have selected the correct model.

We apply this methodology to a large set of factors that the literature has proposed in the last 30 years. We uncover several interesting empirical findings. First, several newly proposed factors (for example, investment, profitability and QMJ) are useful in explaining asset prices, even after accounting for the large set of existing factors proposed up to 2011. Second, the risk price estimates (and the evaluation of the usefulness of factors) that we obtain are stable across different samples, whereas the models selected vary substantially when the data is resampled. Third, applying our test recursively over time would have deemed only a small number of factors proposed in the literature significant. Lastly, we demonstrate how our results differ starkly from the conclusions one would obtain simply by using the risk premia of the factors or the standard Fama-French three factor model as control (as opposed to the model selection procedure we advocate).

Taken together, our results are quite encouraging about the continuing progress of asset pricing research, and suggest that studying the marginal contribution of new factors relative to the vast set of existing ones is a conservative and productive way to screen new factors as they are proposed, as well as to organize the current “zoo of factors.”

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Table 1: Asymptotic Approximation Performance for λ_{useful}

		Bias						RMSE					
		$p = 100$			$p = 300$			$p = 100$			$p = 300$		
T	n	DS	SS	OLS	DS	SS	OLS	DS	SS	OLS	DS	SS	OLS
Panel A: BIC													
360	100	-0.66	-6.29	-	-0.34	-7.14	-	4.90	7.63	-	5.62	8.18	-
360	300	0.19	-5.03	3.20	0.94	-7.36	-	4.45	6.95	6.60	4.93	8.76	-
360	500	0.10	-4.98	2.06	0.41	-7.10	18.87	4.33	7.08	5.57	4.50	8.69	27.28
	480												
480	100	-0.58	-5.21	-	-1.22	-6.42	-	4.18	6.59	-	4.85	7.40	-
480	300	0.14	-4.61	2.55	0.97	-6.22	-	3.80	6.22	5.33	4.19	7.72	-
480	500	0.23	-4.33	1.84	0.50	-6.29	12.94	3.73	6.42	4.66	3.97	7.85	16.30
	600												
600	100	-0.53	-4.33	-	-1.44	-5.95	-	3.74	5.78	-	4.32	6.83	-
600	300	0.21	-4.03	2.15	0.95	-5.59	-	3.41	5.59	4.60	3.75	6.93	-
600	500	0.17	-3.96	1.53	0.61	-5.61	9.98	3.33	5.79	4.07	3.54	7.12	12.17
Panel B: Cross-Validation													
360	100	-0.47	-5.77	-	0.11	-7.94	-	4.88	7.44	-	5.76	8.92	-
360	300	0.15	-5.14	3.25	0.89	-7.37	-	4.38	7.00	6.57	4.90	8.83	-
360	500	0.28	-4.73	2.22	0.28	-7.16	18.13	4.42	7.01	5.70	4.55	8.71	26.72
	480												
480	100	-0.40	-4.67	-	-0.05	-7.03	-	4.18	6.41	-	4.72	7.89	-
480	300	0.24	-4.47	2.66	1.14	-6.26	-	3.82	6.17	5.41	4.24	7.69	-
480	500	0.18	-4.36	1.83	0.53	-6.28	13.03	3.82	6.37	4.80	4.01	7.88	16.48
	600												
600	100	-0.26	-3.95	-	-0.21	-6.53	-	3.70	5.68	-	4.16	7.31	-
600	300	0.19	-3.97	2.14	0.91	-5.69	-	3.44	5.58	4.58	3.71	6.99	-
600	500	0.19	-3.91	1.55	0.55	-5.65	9.81	3.35	5.78	4.08	3.48	7.11	12.09

Note. This table provides the biases and root-mean-squared errors (RMSE) of the estimates of the price of risk λ of the useful factor from Monte Carlo simulations. DS is the double-selection estimator, SS is the single-selection estimator, and OLS is the ordinary least squares without selection. The regularization parameters in the LASSO are selected either by minimizing BIC or using 5-fold cross-validation. The true value λ_{useful} is 16.76. Note that in all cases when $n \leq p$, OLS is infeasible.

Table 2: Asymptotic Approximation Performance for $\lambda_{\text{redundant}}$

		Bias						RMSE					
		$p = 100$			$p = 300$			$p = 100$			$p = 300$		
T	n	DS	SS	OLS	DS	SS	OLS	DS	SS	OLS	DS	SS	OLS
Panel A: BIC													
360	100	0.62	-21.11	-	-2.38	-25.54	-	14.21	28.02	-	16.74	30.37	-
360	300	-0.17	-13.72	-0.23	-0.12	-13.01	-	5.73	19.47	8.89	6.16	17.32	-
360	500	-0.06	-12.01	-0.11	-0.08	-10.27	-1.08	4.32	17.52	5.98	4.37	14.12	26.99
480	100	0.30	-23.46	-	-3.08	-29.92	-	13.58	30.98	-	17.88	34.15	-
480	300	-0.10	-18.07	0.03	-0.04	-16.70	-	5.52	24.25	8.16	5.90	21.50	-
480	500	-0.09	-15.03	-0.18	-0.10	-13.68	-0.46	4.15	21.41	5.46	4.24	18.06	15.42
600	100	0.51	-24.41	-	-3.15	-33.22	-	12.85	33.11	-	18.20	36.84	-
600	300	-0.02	-20.79	-0.03	-0.22	-21.10	-	5.33	27.62	7.61	5.62	26.24	-
600	500	-0.14	-18.20	-0.13	-0.07	-16.99	-0.25	3.98	25.18	5.11	3.99	21.75	12.08
Panel B: Cross-Validation													
360	100	0.52	-16.95	-	-0.97	-21.43	-	12.57	24.64	-	14.65	26.76	-
360	300	-0.23	-14.24	-0.40	-0.04	-12.36	-	5.89	19.97	8.93	6.11	16.55	-
360	500	-0.12	-11.84	-0.17	-0.18	-10.34	-0.44	4.26	17.45	5.91	4.37	14.18	27.01
480	100	0.28	-19.10	-	-1.38	-26.45	-	12.14	27.80	-	14.94	30.82	-
480	300	-0.12	-17.78	-0.27	-0.09	-17.00	-	5.55	24.06	8.13	5.79	21.71	-
480	500	-0.21	-15.19	-0.21	-0.04	-13.60	-0.03	4.13	21.58	5.44	4.21	17.87	15.39
600	100	0.28	-20.60	-	-0.49	-30.10	-	11.78	29.94	-	14.51	33.93	-
600	300	-0.12	-20.46	-0.30	-0.14	-20.73	-	5.33	27.37	7.63	5.70	25.73	-
600	500	-0.15	-17.77	-0.19	-0.04	-16.64	0.11	4.01	24.86	5.12	4.05	21.46	12.26

Note. This table provides the biases and root-mean-squared errors (RMSE) of the estimates of the price of risk λ of the redundant factor from Monte Carlo simulations. DS is the double-selection estimator, SS is the single-selection estimator, and OLS is the ordinary least squares without selection. The regularization parameters in the LASSO are selected either by minimizing BIC or using 5-fold cross-validation. The true value $\lambda_{\text{redundant}}$ is 0. Note that in all cases when $n \leq p$, OLS is infeasible.

Table 3: Asymptotic Approximation Performance for λ_{useless}

		Bias						RMSE					
		$p = 100$			$p = 300$			$p = 100$			$p = 300$		
T	n	DS	SS	OLS	DS	SS	OLS	DS	SS	OLS	DS	SS	OLS
Panel A: BIC													
360	100	0.41	-0.23	-	3.70	5.94	-	13.21	15.83	-	16.81	19.36	-
360	300	0.00	-0.05	0.11	0.07	0.61	-	5.89	8.37	9.00	5.92	7.93	-
360	500	0.09	-0.27	0.13	0.01	-0.40	-0.53	4.23	6.78	5.87	4.41	6.11	26.47
480	100	0.24	-0.11	-	5.10	7.67	-	12.60	14.78	-	18.75	21.23	-
480	300	0.10	0.02	0.00	-0.05	0.93	-	5.51	8.39	8.20	5.81	8.17	-
480	500	0.04	-0.32	0.05	0.03	-0.41	0.39	4.03	7.10	5.37	4.20	6.58	14.96
600	100	-0.03	-0.17	-	5.53	8.72	-	11.92	14.58	-	18.95	21.97	-
600	300	0.20	0.17	0.29	-0.12	0.88	-	5.48	8.82	7.65	5.43	8.44	-
600	500	0.01	-0.51	-0.02	-0.04	-0.61	0.05	3.95	7.24	5.06	4.05	6.61	12.19
Panel B: Cross-Validation													
360	100	-0.10	-0.63	-	2.03	4.08	-	12.12	14.06	-	15.10	16.75	-
360	300	0.14	0.07	0.09	0.12	0.55	-	5.68	8.09	9.02	6.00	7.74	-
360	500	-0.09	-0.49	-0.12	0.00	-0.22	0.31	4.21	6.88	5.86	4.35	6.02	26.75
480	100	0.07	-0.13	-	2.64	5.26	-	11.41	13.17	-	15.40	17.55	-
480	300	-0.04	-0.05	-0.21	0.09	0.75	-	5.65	8.51	8.23	5.74	8.11	-
480	500	-0.10	-0.40	-0.15	0.03	-0.50	-0.02	4.14	6.99	5.43	4.20	6.42	15.33
600	100	0.05	-0.05	-	3.19	6.23	-	11.01	12.99	-	15.74	18.27	-
600	300	0.02	0.11	0.14	0.13	1.19	-	5.48	8.72	7.68	5.48	8.50	-
600	500	0.03	-0.58	-0.04	0.09	-0.51	-0.10	4.01	7.15	5.17	4.08	6.59	12.02

Note. This table provides the biases and root-mean-squared errors (RMSE) of the estimates of the price of risk λ of the useless factor from Monte Carlo simulations. DS is the double-selection estimator, SS is the single-selection estimator, and OLS is the ordinary least squares without selection. The regularization parameters in the LASSO are selected either by minimizing BIC or using 5-fold cross-validation. The true value λ_{useless} is 0. Note that in all cases when $n \leq p$, OLS is infeasible.

Table 4: Testing for Factors Introduced in 2011-2016

id	Factor Description	(1)		(2)		(3)		(4)		(5)	
		DS		SS		FF3		No Selection		Avg. Ret.	
		λ_s	tstat	λ_s	tstat	λ_s	tstat	λ_s	tstat	avg.ret.	tstat
		(bp)	(DS)	(bp)	(DS)	(bp)		(bp)		(bp)	
84	Maximum Return	-50	-0.40	-134	-2.93***	15	0.43	-291	-1.03	3	0.16
85	Percent Accruals	-6	-0.25	17	1.36	-18	-1.81*	-34	-0.58	11	1.18
86	Cash Holdings	-102	-1.48	52	1.85*	26	1.24	-143	-0.87	19	1.34
87	HML Devil	44	0.53	29	0.44	-108	-2.27**	-20	-0.10	27	1.55
88	Gross profitability	-22	-0.50	11	0.37	42	2.62***	50	0.49	21	1.71*
89	Organizational Capital	-44	-0.99	-46	-1.67*	46	2.77***	20	0.23	41	2.67***
90	Betting Against Beta	-17	-0.51	-41	-1.61	19	1.02	-22	-0.32	92	5.42***
91	Quality Minus Junk	118	2.84***	21	0.72	40	2.35**	84	0.95	50	4.09***
92	HXZ Investment	41	1.93*	-1	-0.05	-3	-0.27	25	0.58	38	4.15***
93	HXZ Profitability	136	4.00***	15	0.68	34	2.22**	10	0.18	57	4.51***
94	Employee Growth	14	0.36	-20	-0.92	-2	-0.13	-27	-0.32	7	0.63
95	RMW	122	4.66***	-2	-0.09	22	1.62	105	1.46	36	3.08***
96	CMA	64	2.37**	6	0.31	1	0.05	21	0.37	31	3.23***
97	Intermediary Capital	-35	-0.54	-67	-1.00	97	1.86*	92	0.58		
98	Intermediary Investment	53	0.65	-71	-0.92	63	0.98	17	0.10	116	3.57***
99	Convertible Debt	-10	-0.59	-12	-0.90	6	0.61	-60	-1.70*	7	0.90

Note. The table reports tests for the contribution of factors introduced in 2011-2016 relative to the set of factors introduced up to 2010. The set of 248 potential controls includes the factors introduced up to and including 2010, their squares and interactions with size. For each column, we show the estimate of risk price λ_g and the t-statistic. The first column uses the double-selection (DS) method, our benchmark. The regularization parameters in each selection are chosen by 5-fold cross-validation. The second column uses the single-selection (SS) method. The third column uses the Fama-French 3 factors as controls. The fourth column estimates risk prices using all factors as controls, without using dimension-reduction techniques, with simple OLS. The last column reports the risk premium of each tradable factor. Sample period is from July 1980 to December 2016.

Table 5: Testing Factors Recursively by Year of Publication

Year	(1)	(2)	(3)						
	# Assets	# Controls	New factors (IDs)						
1994	450	65	23	24					
1995	500	71	25	26	27				
1996	500	80	28	29					
1997	550	86	<u>30</u>						
1998	575	89	31	32	33	34	35	36	
1999	725	107	37	38					
2000	750	113	39	40	41	42			
2001	800	125	43	<u>44</u>	45				
2002	825	134	46	<u>47</u>	48				
2003	875	143	49	50	51				
2004	925	152	52	<u>53</u>	54	55	56		
2005	1025	167	57	58	59	<u>60</u>	61		
2006	1100	182	62	63	64	65	66	67	
2007	1275	203	<u>69</u>	70	71				
2008	1350	212	<u>72</u>	73	74	75			
2009	1450	224	76	77	78	<u>79</u>			
2010	1525	236	80	81	<u>82</u>	<u>83</u>			
2011	1625	248	84	85					
2012	1675	254	86						
2013	1700	257	87	88	89				
2014	1750	266	90	<u>91</u>	92	<u>93</u>	94		
2015	1825	281	<u>95</u>	96					
2016	1825	287	97	98	99				

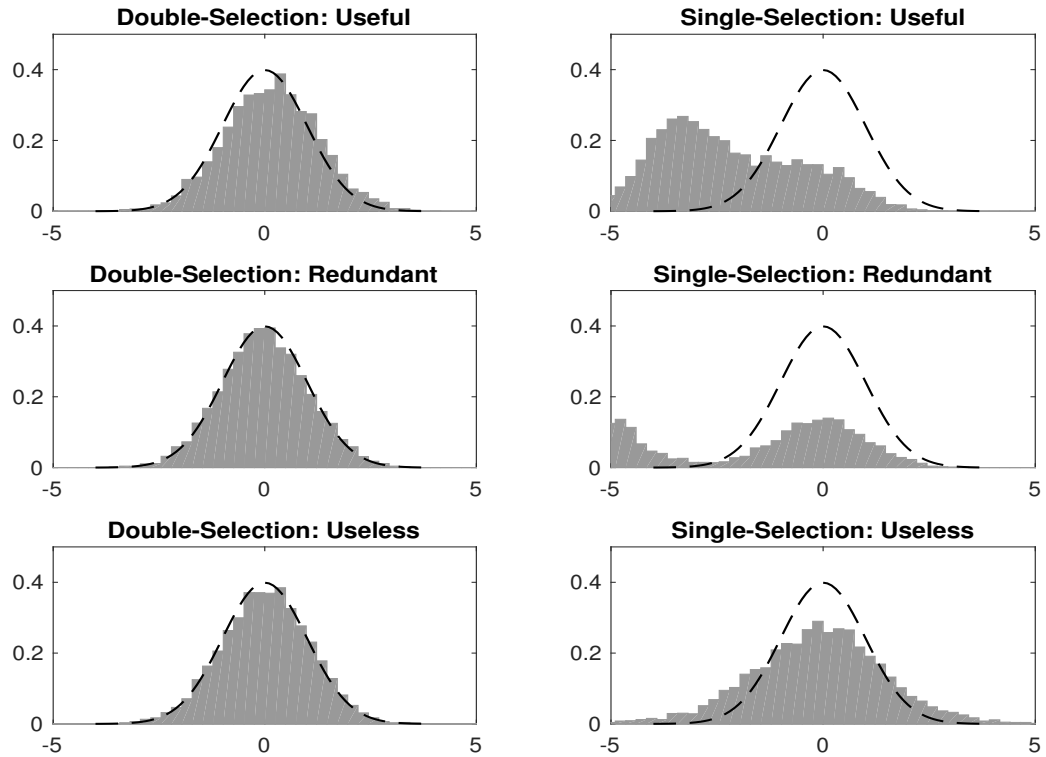
Note. The table reports the results of a recursive factor-testing exercise, from 1994 to 2016. For each year t , column (1) reports the number of test assets available for the test at that point in time, sorted on characteristics available up to then. Column (2) reports the number of controls available in each year t , i.e. the number of potential controls in h_t based on factors introduced up to then, plus their squares and interactions with size. Column (3) shows for each year the IDs of the factors that were introduced during that year. We then test whether each new factor contributes to explaining asset prices relative to the factors published in previous years, using only the data available up to time t . We underline the IDs in column (3) every time the factor appears significant using our double-selection test. The regularization parameters in each selection are chosen by 5-fold cross-validation.

Table 6: Robustness for Factors Introduced in 2011-2016

id	Factor Description	(1)		(2)		(3)		(4)		(5)	
		Bivariate 5×5		Sequential 5×5		Pre-1994		Tradable Factors		Elastic Net	
		λ_s (bp)	tstat (DS)	λ_s (bp)	tstat (DS)	λ_s (bp)	tstat (DS)	λ_s (bp)	tstat (DS)	λ_s (bp)	tstat (DS)
84	Maximum Return	-50	-0.40	-108	-0.86	-167	-1.19	74	0.60	-103	-0.82
85	Percent Accruals	-6	-0.25	-19	-0.81	3	0.12	-38	-1.47	-11	-0.42
86	Cash Holdings	-102	-1.48	-143	-2.00**	-27	-0.40	59	0.73	-99	-1.40
87	HML Devil	44	0.53	-22	-0.28	-35	-0.43	5	0.07	78	1.03
88	Gross profitability	-22	-0.50	-34	-0.77	-41	-0.92	-55	-1.03	-6	-0.11
89	Organizational Capital	-44	-0.99	-32	-0.75	-51	-1.20	-67	-1.52	-49	-1.15
90	Betting Against Beta	-17	-0.51	-25	-0.73	-42	-1.31	16	0.52	-2	-0.06
91	Quality Minus Junk	118	2.84***	107	2.60***	193	4.51***	31	0.81	91	2.28**
92	HXZ Investment	41	1.93*	48	2.28**	64	2.83***	48	2.10**	42	1.85*
93	HXZ Profitability	136	4.00***	121	3.95***	150	4.20***	80	2.43**	64	1.88*
94	Employee Growth	14	0.36	52	1.37	-11	-0.31	10	0.26	-2	-0.05
95	RMW	122	4.66***	117	3.43***	117	2.77***	94	2.56***	111	3.53***
96	CMA	64	2.37**	59	2.27**	74	2.92***	56	2.02**	19	0.70
97	Intermediary Capital	-35	-0.54	-29	-0.48	-16	-0.27	-31	-0.49	-36	-0.47
98	Intermediary Investment	53	0.65	74	0.90	27	0.32	-96	-1.17	-37	-0.40
99	Convertible Debt	-10	-0.59	-8	-0.50	13	0.95	14	1.14	6	0.30

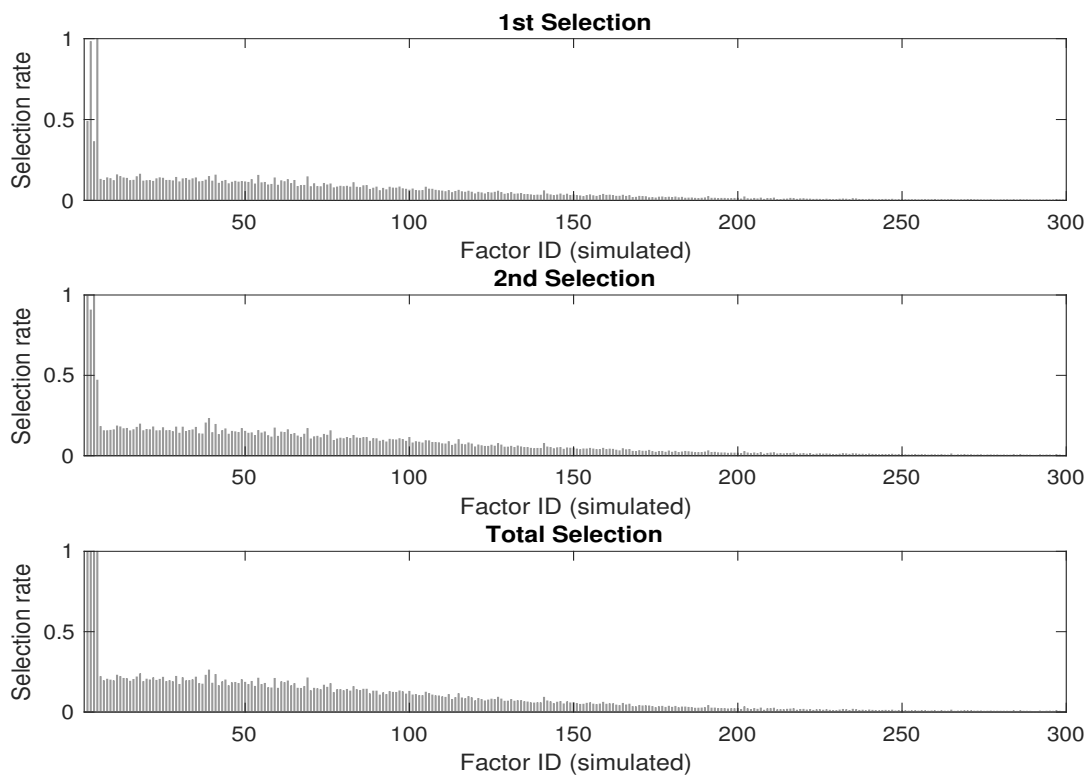
Note. The table reports robustness tests for the estimates of risk prices for factors introduced in 2011-2016 relative to the set of factors introduced up to 2010. The first column shows the same results as in the first column of Table 4 for convenience. The second column shows the results using sequentially-sorted portfolios, the third column uses only characteristics-sorted portfolios available up to 1994, and the fourth column uses only tradable factors. In the last column, we use Elastic Net regularization for factor selection instead of LASSO. The regularization parameters in each selection are chosen by 5-fold cross-validation.

Figure 1: Histograms of the Standardized Estimates in Simulations



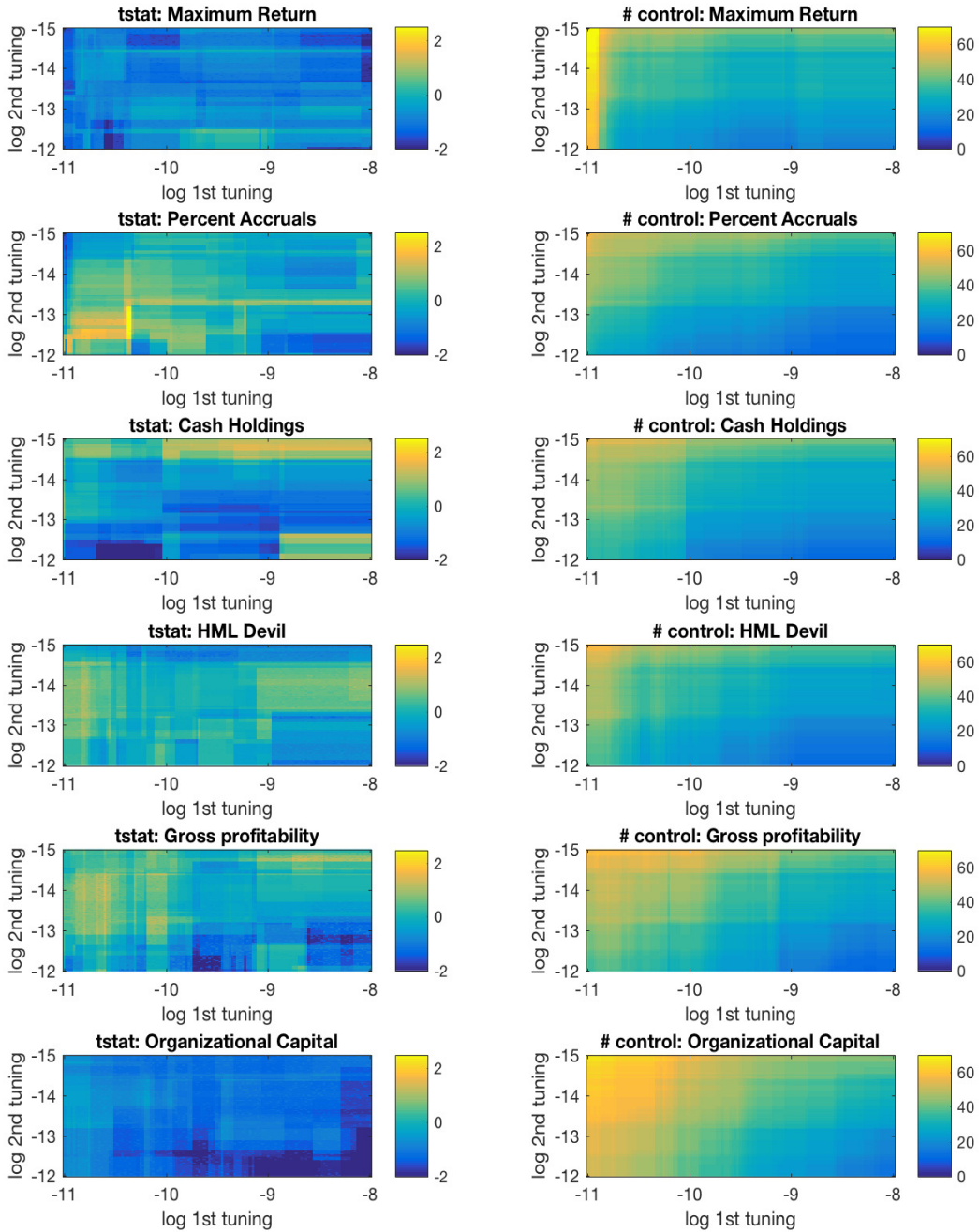
Note. The figure presents the histograms of the standardized double-selection and single-selection estimates using estimated standard errors, compared with the standard normal density in solid dashed lines. The left panel reports the double-selection histograms, and the right panel the single-selection histograms. The top row reports the distribution of standardized estimates for a useful factor; the middle row for a redundant factor; the last row for a useless factor. In the simulation, we set $T = 600$, $n = 500$, and $p = 300$. The regularization parameters in each selection are chosen by 5-fold cross-validation.

Figure 2: Histograms of the Selected Variables

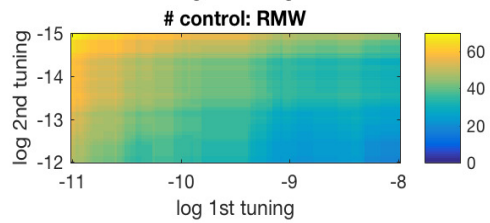
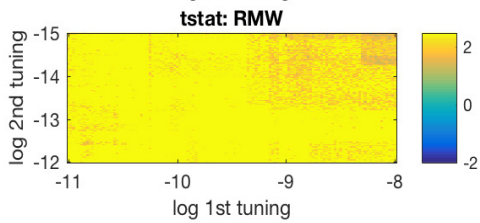
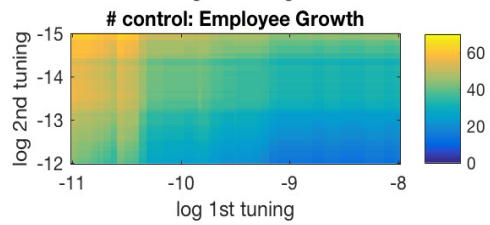
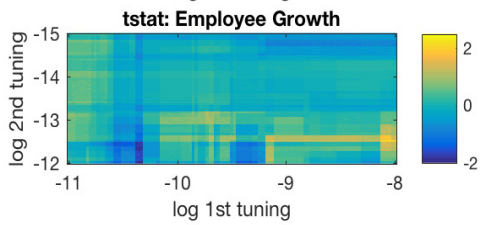
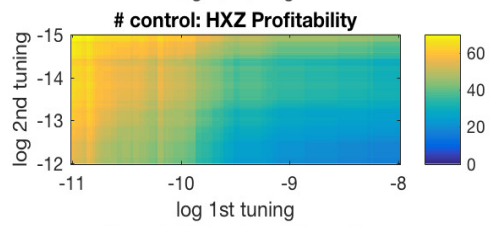
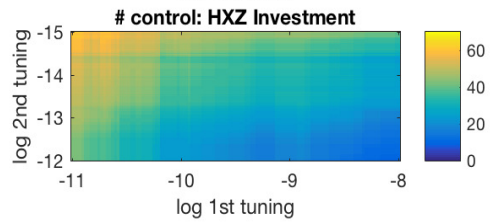
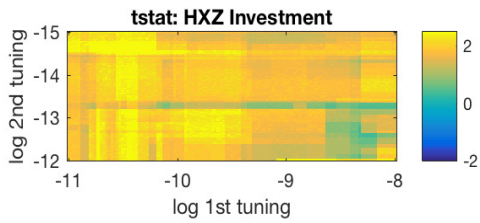
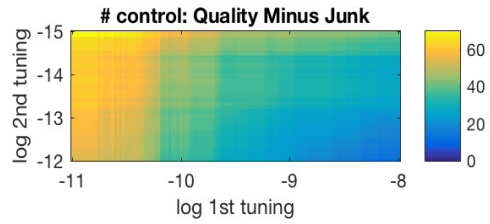
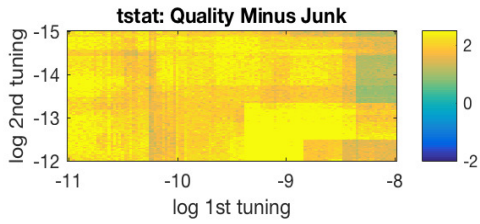
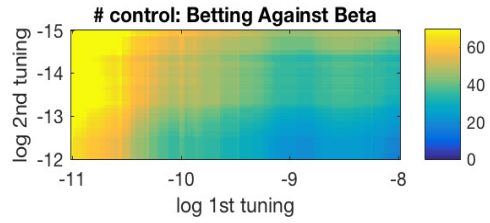
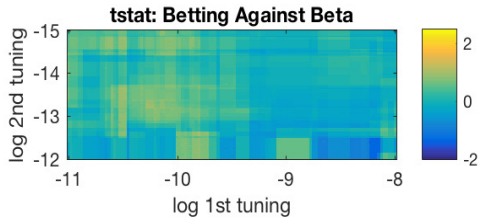


Note. The figure reports how often each factor is selected in each step of our double selection procedure (first and second panels corresponding to the first and second step, and their union in the bottom panel), in Monte Carlo simulations. Each factor corresponds to a number on the X axis. Factors 1 - 4 are part of the true factors in the DGP. Factors 5 - 152 are redundant, whereas factors 153 - 300 are useless. We set $T = 600$, $n = 500$, and $p = 300$. The regularization parameters in each selection are chosen by 5-fold cross-validation.

Figure 3: Factors Introduced in 2011-2016: Robustness to Tuning Parameters



Note. The figures provide heat maps for double-selection tests of factors introduced in 2011-2016, as in the first column of Table 4, using a wide range of tuning parameters, for the first LASSO stage on the X axis and for the second stage on the Y axis. Each factor corresponds to a row. For each factor, the left panel shows the t-statistics from our double selection method; the right panel indicates the number of controls selected from h_t (the union of the 1st and 2nd selections).



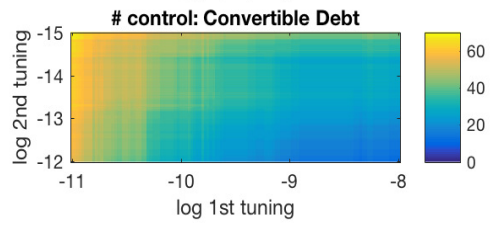
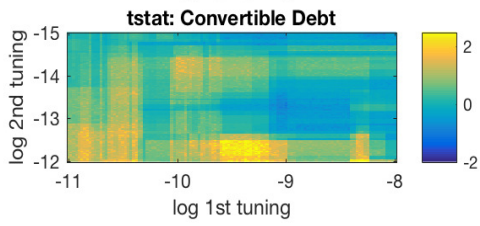
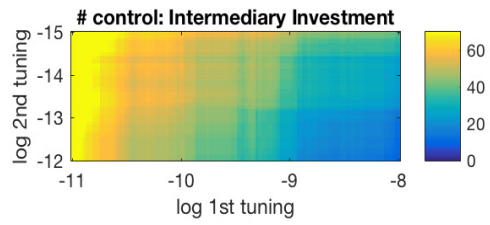
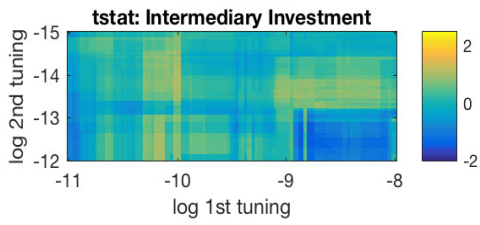
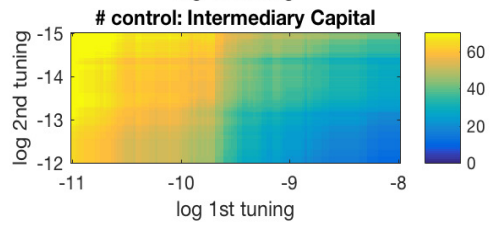
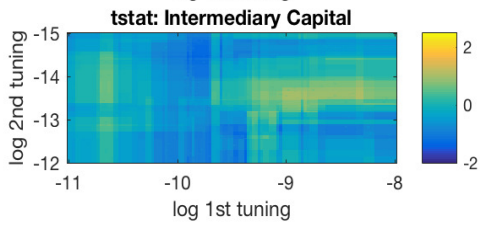
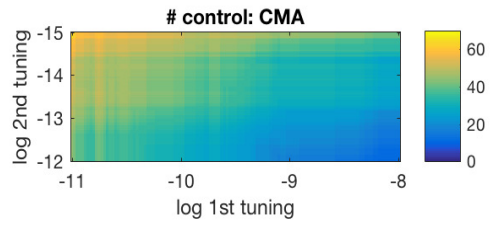
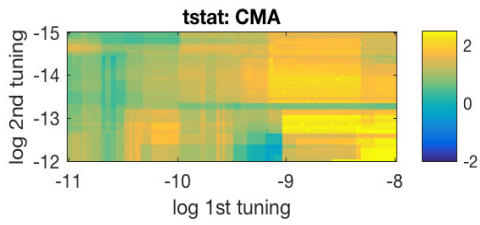
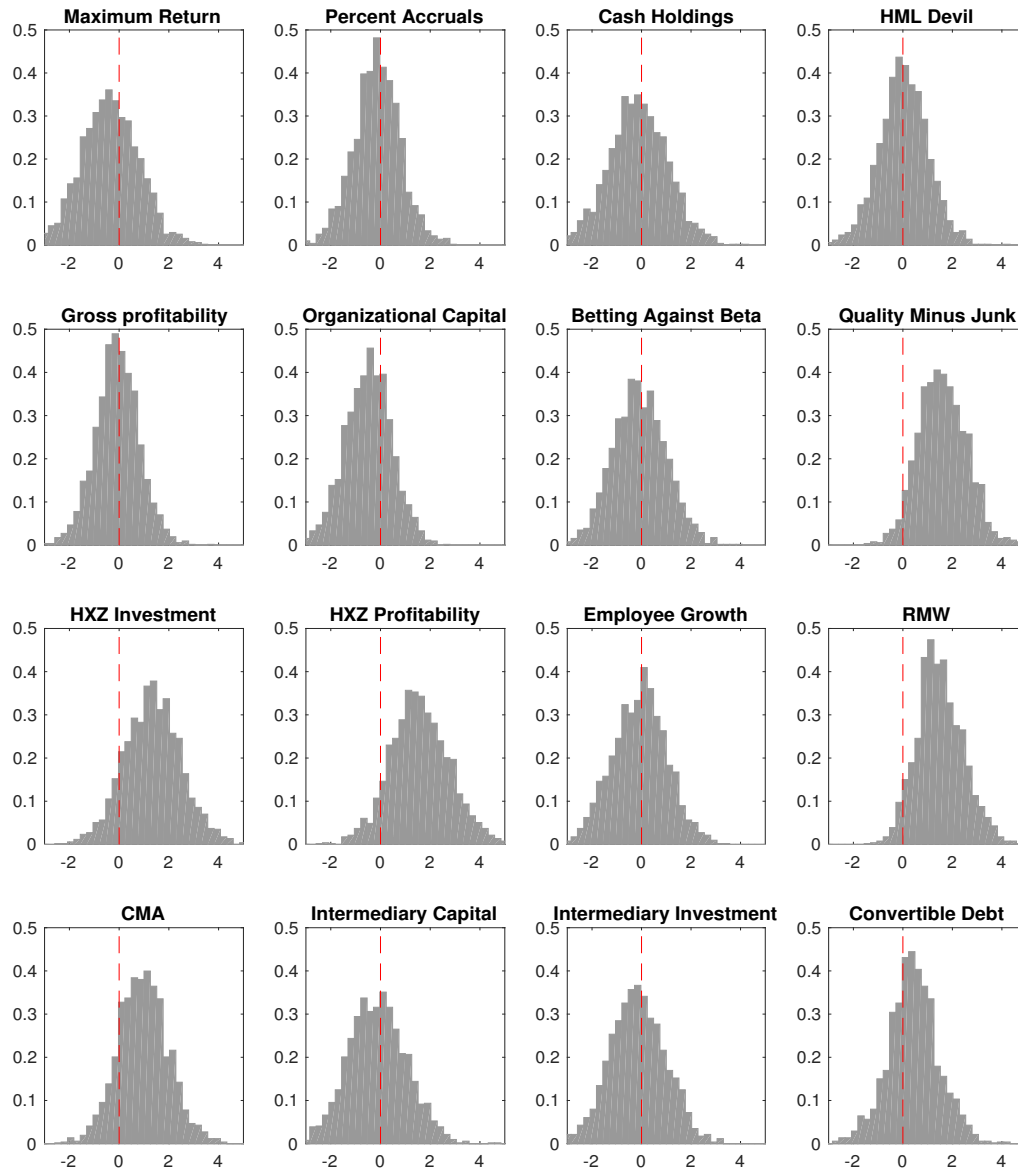
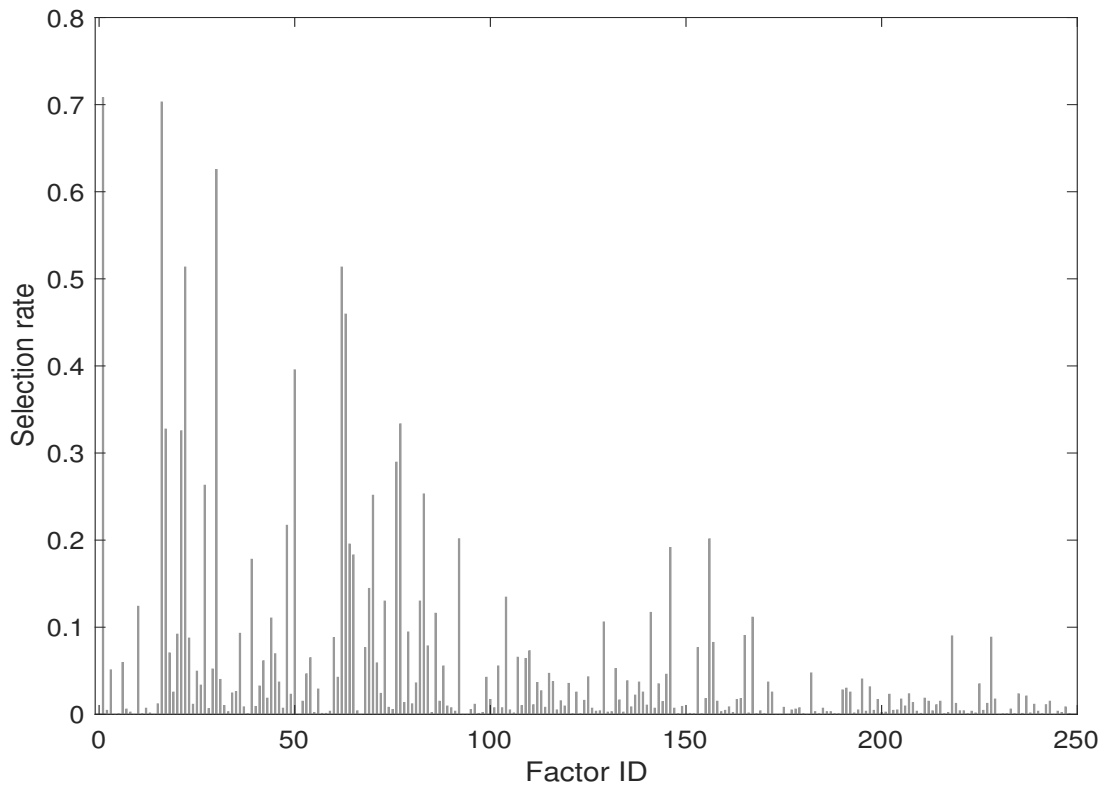


Figure 4: Subsamples: Factor t-statistics



Note. The figure reports the histogram of individual factor t-statistics estimates across 2,000 bootstrap samples, for the factors introduced since 2011 (as in Table 4). We bootstrap our data over both the time-series and the cross-sectional (test assets) dimensions.

Figure 5: Subsamples: Factor Selection Rate



Note. The figure reports the control factor selection rates for the tests of Table 4 (i.e., the factors selected by the first LASSO step of the double-selection procedure), across bootstrap samples selected as in Figure 4. The figure shows, for each factor identified by the factor ID (on the X axis), in what fraction of the 2,000 samples that factor is selected by LASSO. The first 83 factors are the original factors; the next 83 factors are the squared factors and the last 82 are their interaction with Small Minus Big.

Appendix A List of Factors

Table A.1: Factor Zoo

id	Category	Description	Year	Avg.Ret	Annual S.R.	Reference
1	V	Excess Market Return	1965	0.65%	50.61%	Sharpe (1964)
2	T	Beta	1973	-0.18%	-10.71%	Fama and MacBeth (1973)
3	V	Earnings to Price	1977	0.20%	21.79%	Basu (1977)
4	V	Dividend to Price	1982	0.12%	9.02%	Litzenberger and Ramaswamy (1982)
5	M	Unexpected Quarterly Earnings	1982	0.04%	7.90%	Rendleman et al. (1982)
6	V	Long-Term Reversal	1985	0.19%	26.54%	Bondt and Thaler (1985)
7	V	Leverage	1988	0.12%	13.39%	Bhandari (1988)
8	T	Bid-Ask Spread	1989	0.12%	8.27%	Amihud and Mendelson (1989)
9	IN	Cash Flow to Debt	1989	0.06%	12.17%	Ou and Penman (1989)
10	IN	Current Ratio	1989	-0.01%	-0.49%	Ou and Penman (1989)
11	IN	Change in Current Ratio	1989	0.05%	14.00%	Ou and Penman (1989)
12	IN	Change in Quick Ratio	1989	0.03%	6.80%	Ou and Penman (1989)
13	IN	Change Sales-to-Inventory	1989	0.03%	7.28%	Ou and Penman (1989)
14	IN	Quick Ratio	1989	0.00%	0.03%	Ou and Penman (1989)
15	IN	Sales to Cash	1989	-0.04%	-4.72%	Ou and Penman (1989)
16	IN	Sales to Inventory	1989	-0.01%	-1.65%	Ou and Penman (1989)
17	IN	Sales to Receivables	1989	0.15%	22.42%	Ou and Penman (1989)
18	V	Depreciation / PP&E	1992	0.11%	10.83%	Holthausen and Larcker (1992)
19	V	Change in Depreciation	1992	0.05%	11.90%	Holthausen and Larcker (1992)
20	T	Small Minus Big	1993	0.12%	14.11%	Fama and French (1993)
21	V	High Minus Low	1993	0.32%	37.11%	Fama and French (1993)
22	T	Short-Term Reversal	1993	0.29%	29.95%	Jegadeesh and Titman (1993)
23	P	Industry-Adjusted Change in Employees	1994	-0.01%	-2.71%	Asness et al. (2000)

Note. The factor zoo contains 98 tradable and 1 nontradable (No.97) factors for monthly data from July 1980 to December 2016. In addition to these publicly available factors, we follow [Fama and French \(1993\)](#) to construct value-weighted portfolios as factors using firm characteristics collected in [Green et al. \(2016\)](#). For each characteristic therein, we sort all stocks into deciles based on their previous year-end values, then build and rebalance a long-short portfolio (top 30% - bottom 30% or 1-0 dummy difference) every June. For factor classification, “M” is Momentum, “V” is Value-versus-Growth, “I” is Investment, “P” is Profitability, “IN” is Intangibles, and “T” is Trading Frictions.

24	P	Sales Growth	1994	-0.06%	-8.34%	Lakonishok et al. (1994)
25	V	Dividend Initiation	1995	0.16%	14.32%	Michaely et al. (1995)
26	V	Dividend Omission	1995	0.21%	17.93%	Michaely et al. (1995)
27	T	New Equity Issue	1995	0.15%	13.46%	Loughran and Ritter (1995)
28	I	Working Capital Accruals	1996	0.13%	25.75%	Sloan (1996)
29	V	Sales to Price	1996	0.27%	35.26%	Barbee Jr et al. (1996)
30	M	Momentum	1997	0.57%	43.95%	Carhart (1997)
31	IN	Industry Adjusted Change in Capital Expenditures	1998	0.10%	22.13%	Abarbanell and Bushee (1998)
32	IN	Change in Gross Margin - Change in Sales	1998	0.02%	4.32%	Abarbanell and Bushee (1998)
33	IN	Change in Sales - Change in Inventory	1998	0.04%	8.63%	Abarbanell and Bushee (1998)
34	IN	Change in Sales - Change in A/R	1998	0.19%	47.59%	Abarbanell and Bushee (1998)
35	IN	Change in Sales - Change in SG&A	1998	-0.13%	-26.65%	Abarbanell and Bushee (1998)
36	T	Share Turnover	1998	-0.05%	-4.40%	Datar et al. (1998)
37	M	Industry Momentum	1999	-0.13%	-13.23%	Moskowitz and Grinblatt (1999)
38	M	Number of Earnings Increases	1999	-0.06%	-15.80%	Barth et al. (1999)
39	V	Industry-Adjusted Book to Market	2000	0.06%	9.91%	Asness et al. (2000)
40	V	Industry-Adjusted Cash Flow to Price Ratio	2000	0.17%	27.98%	Asness et al. (2000)
41	T	Industry-Adjusted Size	2000	0.03%	4.59%	Asness et al. (2000)
42	P	Financial Statements Score	2000	-0.01%	-1.53%	Piotroski (2000)
43	T	Dollar Trading Volume	2001	0.23%	25.75%	Chordia et al. (2001)
44	T	Volatility for Dollar Trading Volume	2001	0.14%	19.78%	Chordia et al. (2001)
45	T	Volatility for Share Turnover	2001	0.03%	2.12%	Chordia et al. (2001)
46	I	Change in Inventory	2002	0.13%	24.79%	Thomas and Zhang (2002)
47	I	Change in Tax Expense	2011	-0.02%	-3.41%	Thomas and Zhang (2002)
48	T	Illiquidity	2002	0.19%	19.94%	Amihud (2002)
49	T	Liquidity	2003	0.47%	47.00%	Pástor and Stambaugh (2003b)
50	I	Growth in Long Term Net Operating Assets	2003	0.17%	36.27%	Fairfield et al. (2003)
51	T	Idiosyncratic Return Volatility	2003	0.04%	2.43%	Ali et al. (2003)
52	V	Cash Flow to Price Ratio	2004	0.22%	25.03%	Desai et al. (2004)
53	I	Corporate Investment	2004	0.14%	30.45%	Titman et al. (2004)
54	I	R&D Increase	2004	0.04%	7.89%	Eberhart et al. (2004)
55	IN	Earnings Volatility	2004	0.01%	1.42%	Francis et al. (2004)
56	V	Tax Income to Book Income	2004	0.11%	20.26%	Lev and Nissim (2004)
57	IN	# Years Since First Compustat Coverage	2005	0.06%	6.09%	Jiang et al. (2005)
58	I	Growth in Common Shareholder Equity	2005	-0.15%	-23.68%	Richardson et al. (2005)
59	I	Growth in Long-Term Debt	2005	0.06%	10.95%	Richardson et al. (2005)
60	P	Financial Statement Performance	2005	0.13%	27.16%	Mohanram (2005)
61	IN	Price Delay	2005	0.14%	25.89%	Hou and Moskowitz (2005)
62	M	Change in 6-Month Momentum	2006	0.21%	25.23%	Gettleman and Marks (2006)

63	I	Growth in Capital Expenditures	2006	0.09%	16.10%	Anderson and Garcia-Feijoo (2006)
64	IN	Industry Sales Concentration	2006	-0.02%	-2.57%	Hou and Kimmel (2006)
65	I	R&D to Market Capitalization	2006	0.37%	40.73%	Guo et al. (2006)
66	I	R&D to Sales	2006	0.08%	7.67%	Guo et al. (2006)
67	T	Return Volatility	2006	0.08%	5.72%	Ang et al. (2006)
68	T	Zero Trading Days	2006	-0.03%	-2.11%	Liu (2006)
69	M	Abnormal Earnings Announcement Volume	2007	0.07%	10.16%	Lerman et al. (2008)
70	I	Return on Invested Capital	2007	0.17%	26.53%	Brown and Rowe (2007)
71	IN	Debt Capacity / Firm Tangibility	2007	0.01%	1.31%	Almeida and Campello (2007)
72	P	Industry-Adjusted Change in Asset Turnover	2008	0.08%	21.65%	Soliman (2008)
73	I	Change in Shares Outstanding	2008	0.17%	26.20%	Pontiff and Woodgate (2008)
74	P	Industry-Adjusted Change in Profit Margin	2008	0.11%	19.12%	Soliman (2008)
75	M	Earnings Announcement Return	2008	-0.01%	-2.48%	Brandt et al. (2008)
76	V	Cash Productivity	2009	0.31%	36.72%	Chandrashekar and Rao (2009)
77	M	Revenue Surprise	2009	0.13%	19.19%	Kama (2009)
78	T	Sin Stocks	2009	0.40%	37.15%	Hong and Kacperczyk (2009)
79	T	Cash Flow Volatility	2009	0.14%	21.22%	Huang (2009)
80	IN	Absolute Accruals	2010	0.05%	8.82%	Bandyopadhyay et al. (2010)
81	IN	Capital Expenditures and Inventory	2010	0.17%	29.95%	Chen and Zhang (2010)
82	P	Return on Assets	2010	0.13%	19.68%	Balakrishnan et al. (2010)
83	T	Accrual Volatility	2010	0.12%	19.36%	Bandyopadhyay et al. (2010)
84	T	Maximum Daily Return	2011	0.03%	2.71%	Bali et al. (2011)
85	IN	Percent Accruals	2011	0.11%	19.54%	Hafzalla et al. (2011)
86	IN	Cash Holdings	2012	0.19%	22.22%	Palazzo (2012)
87	V	HML_Devil	2013	0.27%	25.71%	Asness and Frazzini (2013)
88	P	Gross Profitability	2013	0.21%	28.32%	Novy-Marx (2013)
89	IN	Organizational Capital	2013	0.41%	44.25%	Eisfeldt and Papanikolaou (2013)
90	T	Betting Against Beta	2014	0.92%	89.71%	Frazzini and Pedersen (2014)
91	V	Quality Minus Junk	2014	0.50%	67.63%	Asness et al. (2014)
92	I	HXZ_Investment	2014	0.38%	68.72%	Hou et al. (2014)
93	P	HXZ_Profitability	2014	0.57%	74.64%	Hou et al. (2014)
94	I	Employee Growth Rate	2014	0.07%	10.40%	Belo et al. (2014)
95	P	Robust Minus Weak	2015	0.36%	50.97%	Fama and French (2015)
96	I	Conservative Minus Aggressive	2015	0.31%	53.52%	Fama and French (2015)
97	T	Intermediary Capital Risk	2016			He et al. (2016)
98	T	Intermediary Investment	2016	1.16%	59.07%	He et al. (2016)
99	IN	Convertible Debt Indicator	2016	0.07%	14.82%	Valta (2016)

Appendix B Simulation Details

We simulate returns of test assets and factors according to the following steps:

- (1) Simulate C_{e_1} ($n \times 1$) and C_{h_1} ($n \times 4$) independently from multivariate normal distributions. They correspond to the first column of C_e and the first 4 columns of C_h , respectively.
- (2) Calculate the first column of C_g , i.e., C_{g_1} , from C_{e_1} and C_{h_1} using $C_{g_1} = \iota_n \xi_1 + C_{h_1} \chi_1^\top + C_{e_1}$, where χ_1 is a 1×4 matrix.
- (3) Calculate the first column of C_z , i.e., C_{z_1} , using $C_{z_1} = C_{g_1} - C_{h_1} \eta_1^\top$, as implied from the DGP $g_{1t} = \eta_1 h_{1t} + z_{1t}$ we aim to simulate, where η_1 is a 1×4 matrix.
- (4) Calculate $E(r_t)$ using $E(r_t) = \iota_n \gamma_0 + C_{g_1} \lambda_{g_1} + C_{h_1} \lambda_{h_1}$, where λ_{g_1} is a scalar and λ_{h_1} is a 4×1 vector.
- (5) Calculate $\beta_{g_1} = C_{z_1} \Sigma_{z_1}^{-1}$ and $\beta_{h_1} = C_{h_1} \Sigma_{h_1}^{-1} - \beta_{g_1} \eta_1$, as implied from the DGP of r_t we aim to simulate: $r_t = E(r_t) + \beta_{g_1} g_{1t} + \beta_{h_1} h_{1t} + u_t$.
- (6) Generate u_t from a Student's t distribution with 5 degrees of freedom and a covariance matrix Σ_u . Generate $h_{1t} \sim \mathcal{N}_4(0, \Sigma_{h_1})$, and calculate g_{1t} and then r_t using the DGPs specified in Steps (3) and (5), respectively.
- (7) For useless and redundant factors in $\{g_2, h_2\}$, we calculate $(C_{g_2} : C_{h_2}) = \iota_n \theta_0 + (C_{g_1} : C_{h_1}) \theta_1 + C_\epsilon$, where θ_0 is a $1 \times (p-2)$ matrix, θ_1 is a $5 \times (p-2)$ matrix, and C_ϵ is simulated independently from a multivariate normal distribution. A factor in $(C_{g_2} : C_{h_2})$ is either useless or redundant depending on whether the corresponding column of θ_1 is a zero-vector.
- (8) Simulate $(g_{2t}^\top : h_{2t}^\top)^\top = \phi^\top (r_t - E(r_t)) + \nu_t$, where $\phi = \Sigma_r^{-1} (C_{g_2} : C_{h_2})$ is a $n \times (p-2)$ matrix, $\Sigma_r = C_{z_1} \Sigma_{z_1}^{-1} C_{z_1}^\top + C_{h_1} \Sigma_{h_1}^{-1} C_{h_1} + \Sigma_u$, and ν_t is simulated independently from a multivariate normal distribution.

As a result, the desired sparsity on χ , η , and λ_h holds, because the DGPs of g_{1t} and g_{2t} are conditionally independent of h_{2t} given h_{1t} ; C_{g_1} and C_{g_2} are conditionally independent of C_{h_2} given C_{h_1} ; and $E(r_t)$ is conditionally independent of C_{g_2} and C_{h_2} , given C_{g_1} and C_{h_1} .

Online Appendix A Technical Details and Proofs

A.1 Notation

We summarize the notation used throughout. Let e_i be a vector with 1 in the i th entry and 0 elsewhere, whose dimension depends on the context. Let ι_k denote a k -dimensional vector with all entries being 1. We use $a \vee b$ to denote the max of a and b , and $a \wedge b$ as their min for any scalars a and b . We also use the notation $a \lesssim b$ to denote $a \leq Kb$ for some constant $K > 0$; and $a \lesssim_p b$ to denote $a = O_p(b)$. For any time series of vectors $\{a_t\}_{t=1}^T$, we denote $\bar{a} = T^{-1} \sum_{t=1}^T a_t$. In addition, we write $\bar{a}_t = a_t - \bar{a}$. We use the capital letter A to denote the matrix $(a_1 : a_2 : \dots : a_T)$, and write $\bar{A} = A - \iota_T \bar{a}$ correspondingly. We use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ to denote the minimum and maximum eigenvalues of A . We use $\|A\|_1$, $\|A\|_\infty$, $\|A\|$, and $\|A\|_F$ to denote the \mathbb{L}_1 norm, the \mathbb{L}_∞ norm, the operator norm (or \mathbb{L}_2 norm), and the Frobenius norm of a matrix $A = (a_{ij})$, that is, $\max_j \sum_i |a_{ij}|$, $\max_i \sum_j |a_{ij}|$, $\sqrt{\lambda_{\max}(A^\top A)}$, and $\sqrt{\text{Tr}(A^\top A)}$, respectively. We also use $\|A\|_{\text{MAX}} = \max_{i,j} |a_{ij}|$ to denote the \mathbb{L}_∞ norm of A on the vector space. When a is a vector, both $\|a\|$ and $\|a\|_F$ are equal to its Euclidean norm. We use $\|a\|_0$ to denote $\sum_i 1_{\{a_i \neq 0\}}$. We also denote $\text{Supp}(a) = \{i : a_i \neq 0\}$. We write the projection operator with respect to a matrix A as $\mathbb{P}_A = A(A^\top A)^{-1}A^\top$, and the corresponding annihilator as $\mathbb{M}_A = \mathbb{I} - \mathbb{P}_A$, where \mathbb{I} is the identity matrix whose size depends on the context. For a set of indices I , let $A[I]$ denote a sub-matrix of A , which contains all columns indexed in I .

A.2 Technical Assumptions

Assumption A.1 (Sparsity). $\|\lambda_h\|_0 \leq s$, $\|\chi_{j\cdot}\|_0 \leq s$, $\|\eta_{j\cdot}\|_0 \leq s$, $1 \leq j \leq d$, for some s such that $sn^{-1} \rightarrow 0$.

Definition 1 (LASSO and Post-LASSO Estimators). *We consider a generic linear regression problem with sparse coefficients:*

$$Y = X\beta + \varepsilon, \quad \text{subject to} \quad \|\beta\|_0 \leq s,$$

where Y is a $n \times 1$ vector, X is a $n \times p$ matrix, β is $p \times 1$ vector of parameters. We define the LASSO estimator as

$$\bar{\beta} = \arg \min_{\beta} \left\{ n^{-1} \|Y - X\beta\|^2 + n^{-1} \tau \|\beta\|_1 \right\}.$$

We define the Post-LASSO estimator $\tilde{\beta}_{\hat{I}}$ as

$$\tilde{\beta}_{\hat{I}} = \arg \min_{\beta} \left\{ n^{-1} \|Y - X\beta\|^2 : \beta_j = 0, \quad j \notin \hat{I} \right\},$$

where \hat{I} is the set of indices of variables selected by a first-step LASSO, that is, $\hat{I} = \text{Supp}(\bar{\beta})$.

We adopt a high-level assumption on the model selection properties of LASSO and the prediction error bounds of the Post-LASSO estimators in (7) and (8). Belloni and Chernozhukov (2013) provide more primitive conditions for these bounds to hold.

Assumption A.2 (Properties of Post-LASSO Estimators). *The Post-LASSO estimators in (7) and (8) satisfy the following properties:*

1. $\widehat{s} = |\widehat{I}_1 \cup \widehat{I}_2| \lesssim_p s$.
2. Moreover, if $\tau_0 \geq 2c \left\| \lambda_g^\top C_e^\top(\iota_n : \widehat{C}_h) \right\|_1$, for some $c > 1$, then

$$n^{-1/2} \left\| \iota_n(\widetilde{\gamma}_{\widehat{I}_1} - \check{\gamma}_0) + \widehat{C}_h(\widetilde{\lambda}_{\widehat{I}_1} - \check{\lambda}_h) \right\| \lesssim_p sT^{-1/2}(\log(n \vee p \vee T))^{1/2} + \tau_0 s^{1/2} n^{-1}, \quad (\text{A.1})$$

where $\check{\gamma}_0 = \gamma_0 + \xi^\top \lambda_g$ and $\check{\lambda}_h = \chi^\top \lambda_g + \lambda_h$ are the true parameter values given in (2) and (6).

If $\tau_j \geq 2c_j \left\| e_j^\top C_e^\top(\iota_n : \widehat{C}_h) \right\|_1$, for some $c_j > 1$ and $j = 1, 2, \dots, d$, then

$$n^{-1/2} \left\| \iota_n(\widetilde{\xi}_{\widehat{I}_2} - \xi)^\top + \widehat{C}_h(\widetilde{\chi}_{\widehat{I}_2} - \chi)^\top \right\| \lesssim_p sT^{-1/2}(\log(n \vee p \vee T))^{1/2} + \|\tau\|_{\text{MAX}} s^{1/2} n^{-1}, \quad (\text{A.2})$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_d)^\top$, ξ and χ are the true parameter values given in (6).

Assumption A.2 gives a probabilistic upper bound on \widehat{s} . The prediction error bounds in (A.1) and (A.2) are more conservative than the standard results, because the regressors here are estimated. We provide a sketch of the proof for (A.1) in Appendix A.4, for which we need the following sparse eigenvalues assumption. The proof of (A.2) is similar and simpler. Our theoretical result below would also hold if other model selection procedures are employed, provided that they obey similar properties in Assumption A.2.

Assumption A.3 (Sparse Eigenvalues). *There exist $K_1, K_2 > 0$ and a sequence $l_n \rightarrow \infty$, such that with probability approaching 1,*

$$K_1 \leq \phi_{\min}(l_n s) \left[n^{-1}(\iota_n : \widehat{C}_h)^\top(\iota_n : \widehat{C}_h) \right] \leq \phi_{\max}(l_n s) \left[n^{-1}(\iota_n : \widehat{C}_h)^\top(\iota_n : \widehat{C}_h) \right] \leq K_2,$$

where we denote

$$\phi_{\min}(k)[A] = \min_{1 \leq \|v\|_0 \leq k} \frac{v^\top A v}{\|v\|^2}, \quad \text{and} \quad \phi_{\max}(k)[A] = \max_{1 \leq \|v\|_0 \leq k} \frac{v^\top A v}{\|v\|^2}.$$

Assumption A.3 resembles one of the sufficient conditions that lead to desirable statistical properties of LASSO, which has been adopted by, e.g., Belloni et al. (2014b). It implies the restricted eigenvalue condition proposed by Bickel et al. (2009).

Assumption A.4 (Large Deviation Bounds). *The stochastic discount factor, the returns, and the factors satisfy*

$$\|\bar{a}\|_{\text{MAX}} \lesssim_p T^{-1/2}(\log(n \vee p \vee T))^{1/2}, \quad \text{where } a \in \{m, v, z, u\}. \quad (\text{A.3})$$

$$\|T^{-1} \bar{A} \bar{B}^\top - \text{Cov}(a_t, b_t)\|_{\text{MAX}} \lesssim_p T^{-1/2}(\log(n \vee p \vee T))^{1/2}, \quad \text{where } A, B \in \{M, V, Z, U\}. \quad (\text{A.4})$$

Assumption A.4 imposes high-level assumptions on the large deviation type bounds, which can be verified using the same arguments as in Fan et al. (2011) under stationarity, ergodicity, strong mixing, and exponential-type tail conditions.

Next, we impose additional uniform bounds that impose restrictions on the cross-sectional dependence of the “residuals” in the covariance projection (6). Similar assumptions on factor loadings are employed by Giglio and Xiu (2016).

Assumption A.5 (“Moment” Conditions). *The following restrictions hold:*

$$\|C_e\|_{\text{MAX}} \lesssim 1, \quad \|C_e^\top \iota_n\|_{\text{MAX}} \lesssim n^{1/2}, \quad \|C_e^\top C_h\|_{\text{MAX}} \lesssim n^{1/2}, \quad (\text{A.5})$$

$$\|C_e^\top \bar{u}\|_{\text{MAX}} \lesssim_p n^{1/2} T^{-1/2}, \quad \|C_e^\top \bar{U} \bar{V}^\top\|_{\text{MAX}} \lesssim_p n^{1/2} T^{1/2}, \quad (\text{A.6})$$

$$\lambda_{\min}(n^{-1} C_e^\top C_e) \geq K, \quad \|C_e^\top (\beta_g \eta + \beta_h)\|_{\infty} \lesssim s n^{1/2}, \quad \|\beta_h\|_{\infty} \lesssim s. \quad (\text{A.7})$$

In addition, for $a \in \{m, v, z, u\}$, it holds that

$$\|\Sigma_a\|_{\text{MAX}} \lesssim 1, \quad \|C_a\|_{\text{MAX}} \lesssim 1. \quad (\text{A.8})$$

Finally, we impose a joint central limit theorem for $(z_t, \lambda^\top v_t z_t) = (z_t, (1 - \gamma_0 m_t) z_t)$. This can be verified by the standard central limit theory for dependent stochastic processes, if more primitive assumptions are satisfied, see, e.g., White (2000).

Assumption A.6 (CLT). *The following results hold as $T \rightarrow \infty$:*

$$T^{1/2} \begin{pmatrix} \bar{z} \\ -T^{-1} \gamma_0 \bar{M} \bar{Z} - \Sigma_z \lambda_g \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{pmatrix} \right),$$

where Π_{11} , Π_{12} , and Π_{22} are given by

$$\begin{aligned} \Pi_{11} &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T \mathbb{E}(z_s z_t^\top), \\ \Pi_{12} &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T \mathbb{E}(\lambda^\top v_s z_s z_t^\top), \\ \Pi_{22} &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T \mathbb{E}(\lambda^\top v_s \lambda^\top v_t z_s z_t^\top). \end{aligned}$$

Assumption A.7 (Selection for the Asymptotic Variance Estimator). *The Post-LASSO estimator $\tilde{\eta}_{\bar{\tau}}$ satisfies the usual bounds. That is, if $\bar{\tau}_j \geq 2\bar{c}_j \|HZ\|_{\infty}$, for some $\bar{c}_j > 1$, $j = 1, 2, \dots, d$, then we have*

$$\|(\tilde{\eta}_{\bar{\tau}} - \eta)H\| \lesssim_p s^{1/2} (\log(p \vee T))^{1/2}, \quad \text{and} \quad \|\tilde{\eta}_{\bar{\tau}} - \eta\| \lesssim_p s^{1/2} T^{-1/2} (\log(p \vee T))^{1/2}.$$

A.3 Proof of Main Theorems

Proof of Theorem 1. The estimator of λ_g can be written in closed-form as

$$\widehat{\lambda}_g = \left(\widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_g \right)^{-1} \left(\widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \bar{r} \right). \quad (\text{A.9})$$

Moreover, by (2) and (5), we can relate C_g and C_h to β_g and β_h :

$$C_g = C_h \eta^\top + C_z, \quad \text{where} \quad C_h = (\beta_g \eta + \beta_h) \Sigma_h, \quad C_z = \beta_g \Sigma_z. \quad (\text{A.10})$$

Using (3), (5), (A.10), and the fact that

$$\begin{aligned} \widehat{C}_g - C_g &= (\widehat{C}_h - C_h) \eta^\top + (\widehat{C}_z - C_z), \\ \widehat{C}_z - C_z &= \beta_g (T^{-1} \bar{Z} \bar{Z}^\top - \Sigma_z) + T^{-1} \bar{U} \bar{Z}^\top + T^{-1} (\beta_g \eta + \beta_h) \bar{H} \bar{Z}^\top, \\ \widehat{C}_h - C_h &= (\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{H}^\top - \Sigma_h) + T^{-1} \bar{U} \bar{H}^\top + T^{-1} \beta_g \bar{Z} \bar{H}^\top, \end{aligned}$$

we obtain the following decomposition:

$$\begin{aligned} & T^{1/2} (\widehat{\lambda}_g - \lambda_g) \\ &= \left(n^{-1} \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_g \right)^{-1} n^{-1} T^{1/2} \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \left((C_g - \widehat{C}_g) \lambda_g + C_h \lambda_h + \beta_g \bar{z} + ((\beta_g \eta + \beta_h) \bar{h} + \bar{u}) \right) \\ &= T^{1/2} \Sigma_z^{-1} \left(\bar{z} - (T^{-1} \bar{Z} \bar{V}^\top \lambda - \Sigma_z \lambda_g) \right) \\ &\quad + \left(n^{-1} \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_g \right)^{-1} \left(n^{-1} T^{1/2} \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right. \\ &\quad + n^{-1} T^{1/2} \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\beta_g - \widehat{C}_g \Sigma_z^{-1}) \times (\bar{z} - (T^{-1} \bar{Z} \bar{V}^\top \lambda - \Sigma_z \lambda_g)) \\ &\quad - n^{-1} T^{1/2} \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h (\eta^\top \lambda_g + \lambda_h) - \bar{h}) \\ &\quad \left. + n^{-1} T^{1/2} \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right). \end{aligned}$$

We first analyze the leading term. Note that $\gamma_0 \bar{M} = -\bar{V}^\top \lambda$, by Assumption A.6 and applying the Delta method, we have

$$\begin{aligned} & T^{1/2} \left(\Sigma_z^{-1} \bar{z} - \Sigma_z^{-1} (-T^{-1} \gamma_0 \bar{Z} \bar{M} - \Sigma_z \lambda_g) \right) \\ & \xrightarrow{\mathcal{L}} \mathcal{N} \left(0, \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T \mathbb{E} \left((1 - \lambda^\top v_t) (1 - \lambda^\top v_s) \Sigma_z^{-1} z_t z_s^\top \Sigma_z^{-1} \right) \right). \quad (\text{A.11}) \end{aligned}$$

Next, we show that the reminder terms are of a smaller order. By (A.42), we have

$$n^{-1} T^{1/2} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \lesssim_p s(n^{-1/2} + T^{-1/2}) \log(n \vee p \vee T).$$

By (A.27), we have

$$n^{-1} T^{1/2} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \lesssim_p s^2(n^{-1} T^{1/2} + T^{-1/2}) \log(n \vee p \vee T).$$

By (A.40), we have

$$\begin{aligned} & n^{-1}T^{1/2} \left\| \widehat{C}_g^{\text{T}} \mathbb{M}_{(\iota_n; \widehat{C}_h[\widehat{\Gamma}])}(\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^{\text{T}} \lambda - \Sigma_h(\eta^{\text{T}} \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \lesssim p s^2 (n^{-1/2} + T^{-1/2}) \log(n \vee p \vee T). \end{aligned}$$

By Assumption A.4, (A.11), and (A.35), we have

$$\begin{aligned} & n^{-1}T^{1/2} \left\| \widehat{C}_g^{\text{T}} \mathbb{M}_{(\iota_n; \widehat{C}_h[\widehat{\Gamma}])}(\beta_g - \widehat{C}_z \Sigma_z^{-1}) (\bar{z} - (T^{-1} \bar{Z} \bar{V}^{\text{T}} \lambda - \Sigma_z \lambda_g)) \right\| \\ & \leq n^{-1}T^{1/2} \left\| \widehat{C}_g^{\text{T}} \mathbb{M}_{(\iota_n; \widehat{C}_h[\widehat{\Gamma}])}(\beta_g - \widehat{C}_z \Sigma_z^{-1}) \right\| \left\| \bar{z} - (T^{-1} \bar{Z} \bar{V}^{\text{T}} \lambda - \Sigma_z \lambda_g) \right\| \\ & \lesssim p s (n^{-1/2} + T^{-1/2}) \log(n \vee p \vee T). \end{aligned}$$

This concludes the proof. \square

Proof of Theorem 2. By the identical argument in the proof of Theorem 2 of Newey and West (1987), we have

$$\frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr} (1 - \lambda^{\text{T}} v_t) (1 - \lambda^{\text{T}} v_r) (z_t z_r^{\text{T}} + z_r z_t^{\text{T}}) \xrightarrow{p} \Sigma_z \Pi \Sigma_z.$$

So applying the continuous mapping theorem, it is sufficient to show that

$$\widehat{\Sigma}_z \xrightarrow{p} \Sigma_z, \tag{A.12}$$

$$\widetilde{\Pi} - \frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr} (1 - \lambda^{\text{T}} v_t) (1 - \lambda^{\text{T}} v_r) (z_t z_r^{\text{T}} + z_r z_t^{\text{T}}) \xrightarrow{p} 0, \tag{A.13}$$

where

$$Q_{tr} = \left(1 - \frac{|r-t|}{q+1} \right) 1_{\{|t-r| \leq q\}}, \quad \widetilde{\Pi} = \widehat{\Sigma}_z \widehat{\Pi} \widehat{\Sigma}_z.$$

To prove (A.15), we note that by Assumptions A.4 and A.7, we have

$$\begin{aligned} & \left\| \widehat{\Sigma}_z - \Sigma_z \right\|_{\text{MAX}} \\ & \lesssim T^{-1/2} \left\| (\widetilde{\eta}_{\widehat{\Gamma}} - \eta) H \right\| \|Z\|_{\text{MAX}} + T^{-1} \left\| (\widetilde{\eta}_{\widehat{\Gamma}} - \eta) H \right\|^2 + \left\| T^{-1} Z Z^{\text{T}} - \Sigma_z \right\|_{\text{MAX}} \\ & \lesssim p s^{1/2} T^{-1/2} (\log(p \vee T))^{1/2} \|Z\|_{\text{MAX}} + s T^{-1} \log(p \vee T) + T^{-1/2} (\log(n \vee p \vee T))^{1/2} \\ & = o_p(1). \end{aligned} \tag{A.14}$$

As to (A.13), we can decompose its left-hand side as

$$\frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr} (\widehat{\lambda} - \lambda)^{\text{T}} v_t (1 - \widehat{\lambda}^{\text{T}} v_r) (\widehat{z}_t \widehat{z}_r^{\text{T}} + \widehat{z}_r \widehat{z}_t^{\text{T}}) \tag{A.15}$$

$$+\frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr}(1 - \lambda^\top v_t)(\widehat{\lambda} - \lambda)^\top v_r (\widehat{z}_t \widehat{z}_r^\top + \widehat{z}_r \widehat{z}_t^\top) \quad (\text{A.16})$$

$$+\frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr}(1 - \lambda^\top v_t)(1 - \lambda^\top v_r) ((\widehat{z}_t - z_t) \widehat{z}_r^\top + (\widehat{z}_r - z_r) \widehat{z}_t^\top) \quad (\text{A.17})$$

$$+\frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr}(1 - \lambda^\top v_t)(1 - \lambda^\top v_r) (z_t (\widehat{z}_r - z_r)^\top + z_r (\widehat{z}_t - z_t)^\top). \quad (\text{A.18})$$

Analyzing each of these terms, we can obtain that

$$\begin{aligned} & \left\| \frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr}(\widehat{\lambda} - \lambda)^\top v_t(1 - \widehat{\lambda}^\top v_r) (\widehat{z}_t \widehat{z}_r^\top + \widehat{z}_r \widehat{z}_t^\top) \right\|_{\text{MAX}} \\ & \lesssim qT^{-1} \left\| \widehat{Z} \right\| \left\| \iota_T^\top - \widehat{\lambda}^\top V \right\| \left\| (\widehat{\lambda} - \lambda)^\top V \right\|_{\text{MAX}} \left\| \widehat{Z} \right\|_{\text{MAX}} \lesssim_p qs^{1/2}(T^{-1/2} + n^{-1/2}) \|V\|_{\text{MAX}} \|Z\|_{\text{MAX}}, \\ & \left\| \frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr}(1 - \lambda^\top v_t)(\widehat{\lambda} - \lambda)^\top v_r (\widehat{z}_t \widehat{z}_r^\top + \widehat{z}_r \widehat{z}_t^\top) \right\|_{\text{MAX}} \\ & \lesssim qT^{-1} \left\| \iota_T^\top - \lambda^\top V \right\| \left\| \widehat{Z} \right\| \left\| (\widehat{\lambda} - \lambda)^\top V \right\|_{\text{MAX}} \left\| \widehat{Z} \right\|_{\text{MAX}} \lesssim_p qs^{1/2}(T^{-1/2} + n^{-1/2}) \|V\|_{\text{MAX}} \|Z\|_{\text{MAX}}, \\ & \left\| \frac{1}{T} \sum_{t=1}^T \sum_{r=1}^T Q_{tr}(1 - \lambda^\top v_t)(1 - \lambda^\top v_r) ((\widehat{z}_t - z_t) \widehat{z}_r^\top + (\widehat{z}_r - z_r) \widehat{z}_t^\top) \right\|_{\text{MAX}} \\ & \lesssim qT^{-1} \left\| \iota_T^\top - \lambda^\top V \right\| \left\| (\widehat{\eta} - \eta)H \right\| \left\| \widehat{Z} \right\|_{\text{MAX}} \left\| \iota_T^\top - \lambda^\top V \right\|_{\text{MAX}} \\ & \lesssim_p qs^{3/2}(T^{-1/2} + n^{-1/2}) \|V\|_{\text{MAX}} \|Z\|_{\text{MAX}}, \end{aligned}$$

where we use

$$\begin{aligned} & \left\| \iota_T^\top - \lambda^\top V \right\| \lesssim T^{1/2} + \left\| \bar{M} \right\| + \left\| \lambda^\top \bar{v} \right\| \lesssim_p T^{1/2}, \\ & \left\| \iota_T^\top - \lambda^\top V \right\|_{\text{MAX}} \lesssim 1 + \left\| \lambda^\top V \right\|_{\text{MAX}} \lesssim s \|V\|_{\text{MAX}}, \\ & \left\| \iota_T^\top - \widehat{\lambda}^\top V \right\| \leq \left\| \iota_T^\top - \lambda^\top V \right\| + \left\| (\widehat{\lambda} - \lambda)^\top V \right\| \lesssim_p T^{1/2} + \left\| \widehat{\lambda} - \lambda \right\| \|V\| \lesssim_p T^{1/2}, \\ & \left\| \widehat{Z} \right\| \lesssim T^{1/2} \left\| \widehat{\Sigma}_z \right\|^{1/2} \lesssim_p T^{1/2} \left\| \Sigma_z \right\|^{1/2} \lesssim T^{1/2}, \\ & \left\| (\widehat{\lambda} - \lambda)^\top V \right\|_{\text{MAX}} \leq \left\| \widehat{\lambda} - \lambda \right\|_\infty \|V\|_{\text{MAX}} \leq \left\| \widehat{\lambda} - \lambda \right\| \|V\|_{\text{MAX}} \lesssim_p s^{1/2}(T^{-1/2} + n^{-1/2}) \|V\|_{\text{MAX}}, \\ & \left\| \widehat{Z} \right\|_{\text{MAX}} \leq \left\| (\widehat{\eta} - \eta)H \right\| + \|Z\|_{\text{MAX}} \lesssim_p \|Z\|_{\text{MAX}}, \end{aligned}$$

which hold by (A.14), Assumption A.4, and Lemma 7. This concludes the proof. \square

A.4 Proof of Lemmas

Proof of (A.1). We provide a sketch of the proof, as they are identical to Belloni and Chernozhukov (2013). With respect to the optimization problem (7), we define

$$Q(\gamma, \lambda) = n^{-1} \left\| \bar{r} - \iota_n \gamma - \widehat{C}_h \lambda \right\|^2.$$

We denote the solution to this problem as $\tilde{\gamma}$ and $\tilde{\lambda}$. Let $\delta = \tilde{\lambda} - \check{\lambda}_h$. Note by (5) and (2), we have

$$\mathbb{E}(r_t) = \iota_n \check{\gamma}_0 + C_h \check{\lambda}_h + C_e \lambda_g, \quad \text{and} \quad \bar{r} = \mathbb{E}(r_t) + \beta_g \bar{g} + \beta_h \bar{h} + \bar{u}.$$

By direct calculations, we have

$$\begin{aligned} & Q(\tilde{\gamma}, \tilde{\lambda}) - Q(\check{\gamma}_0, \check{\lambda}_h) - n^{-1} \left\| \iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\|^2 \\ &= -2n^{-1} \left(\bar{r} - \iota_n \check{\gamma}_0 - \widehat{C}_h \check{\lambda}_h \right)^\top \left(\iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right) \\ &= -2n^{-1} \left(\beta_g \bar{g} + \beta_h \bar{h} + \bar{u} + (C_h - \widehat{C}_h) \check{\lambda}_h + C_e \lambda_g \right)^\top \left(\iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right) \\ &\geq -2n^{-1} \left\| \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} + (C_h - \widehat{C}_h) \check{\lambda}_h \right\| \left\| \iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\| \\ &\quad - 2n^{-1} \left\| (C_e \lambda_g)^\top (\iota_n : \widehat{C}_h) \right\|_1 \left\| (\tilde{\gamma} - \check{\gamma}_0 : \delta)^\top \right\|_1 \\ &\geq -2n^{-1} \left\| \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} + (C_h - \widehat{C}_h) \check{\lambda}_h \right\| \left\| \iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\| \\ &\quad - \tau_0 K^{-1} n^{-1} (|\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1 + \|\delta_{I^c}\|_1), \end{aligned}$$

where I is the set of non-zeros in $\check{\lambda}_h$, I^c is its complement, and δ_I is a sub-vector of δ with all entries taken from I .

On the other hand, by definition of $\tilde{\gamma}$ and $\tilde{\lambda}$, we have

$$\begin{aligned} Q(\tilde{\gamma}, \tilde{\lambda}) - Q(\check{\gamma}_0, \check{\lambda}_h) &\leq \tau_0 n^{-1} \left(\left\| (\check{\gamma}_0 : \check{\lambda}_h)^\top \right\|_1 - \left\| (\tilde{\gamma} : \tilde{\lambda})^\top \right\|_1 \right) \\ &\leq \tau_0 n^{-1} (|\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1 - \|\delta_{I^c}\|_1). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & n^{-1} \left\| \iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\|^2 - \tau_0 c^{-1} n^{-1} (|\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1 + \|\delta_{I^c}\|_1) \\ &\quad - 2n^{-1} \left\| \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} + (C_h - \widehat{C}_h) \check{\lambda}_h \right\| \left\| \iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\| \\ &\leq \tau_0 n^{-1} (|\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1 - \|\delta_{I^c}\|_1), \end{aligned} \tag{A.19}$$

where we use the fact that

$$\tau_0 \geq 2c \left\| \lambda_g^\top C_e^\top (\iota_n : \widehat{C}_h) \right\|_1.$$

If it holds that

$$n^{-1} \left\| \iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\| - 2n^{-1} \left\| \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} + (C_h - \widehat{C}_h) \check{\lambda}_h \right\| < 0,$$

we can establish that

$$n^{-1/2} \left\| \iota_n (\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\| \lesssim_p sT^{-1/2} (\log(n \vee p \vee T))^{1/2},$$

where we use the fact that

$$n^{-1/2} \|\beta_g \bar{g}\| \lesssim \|\beta_g\|_{\text{MAX}} \|\bar{g}\|_{\text{MAX}} \lesssim_p T^{-1/2}, \quad (\text{A.20})$$

$$n^{-1/2} \|\bar{u}\| \lesssim \|\bar{u}\|_{\text{MAX}} \lesssim_p T^{-1/2} (\log(n \vee p \vee T))^{1/2}, \quad (\text{A.21})$$

$$n^{-1/2} \|\beta_h \bar{h}\| \leq \|\beta_h\|_{\infty} \|\bar{h}\|_{\text{MAX}} \lesssim_p s T^{-1/2} (\log(n \vee p \vee T))^{1/2}, \quad (\text{A.22})$$

$$n^{-1/2} \left\| (C_h - \widehat{C}_h) \check{\lambda}_h \right\| \lesssim \left\| C_h - \widehat{C}_h \right\|_{\text{MAX}} \left\| \check{\lambda}_h \right\|_1 \lesssim_p s T^{-1/2} (\log(n \vee p \vee T))^{1/2}. \quad (\text{A.23})$$

Otherwise, from (A.19) it follows that

$$-c^{-1} (|\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1 + \|\delta_{I^c}\|_1) \leq |\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1 - \|\delta_{I^c}\|_1,$$

which leads to, writing $\bar{c} = (c+1)(c-1)^{-1}$,

$$\|\delta_{I^c}\| \leq \bar{c} (|\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1).$$

Then by (A.19) again as well as the restricted eigenvalue condition in Belloni and Chernozhukov (2013), we obtain

$$\begin{aligned} & \left\| \iota_n(\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\|^2 - 2 \left\| \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} + (C_h - \widehat{C}_h) \check{\lambda}_h \right\| \left\| \iota_n(\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\| \\ & \leq (1 + c^{-1}) \tau_0 (|\tilde{\gamma} - \check{\gamma}_0| + \|\delta_I\|_1) \lesssim \tau_0 s^{1/2} n^{-1/2} \left\| \iota_n(\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} n^{-1/2} \left\| \iota_n(\tilde{\gamma} - \check{\gamma}_0) + \widehat{C}_h \delta \right\| & \lesssim n^{-1/2} \left\| \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} + (C_h - \widehat{C}_h) \check{\lambda}_h \right\| + \tau_0 s^{1/2} n^{-1} \\ & \lesssim_p s T^{-1/2} (\log(n \vee p \vee T))^{1/2} + \tau_0 s^{1/2} n^{-1}. \end{aligned}$$

The Post-LASSO estimator converges at the same rate following the same arguments as in Belloni and Chernozhukov (2013).

□

Lemma 1. *Under Assumptions A.1, A.2, A.4, A.5, we have*

$$n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| \lesssim_p s (n^{-1/2} + T^{-1/2}) (\log(n \vee p \vee T))^{1/2}. \quad (\text{A.24})$$

$$n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \lesssim_p s (n^{-1/2} + T^{-1/2}) (\log(n \vee p \vee T))^{1/2}. \quad (\text{A.25})$$

Proof of Lemma 1. Using the fact that $\widehat{I}_2 \subseteq \widehat{I}$ and by (A.2), we have

$$\begin{aligned} n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| & = n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\widehat{C}_h \chi^\top + \iota_n \xi^\top) \right\| \leq n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}_2])} (\widehat{C}_h \chi^\top + \iota_n \xi^\top) \right\| \\ & \leq n^{-1/2} \left\| \iota_n (\xi - \tilde{\xi}_{\widehat{I}_2})^\top + \widehat{C}_h \chi^\top - \widehat{C}_h \tilde{\chi}_{\widehat{I}_2}^\top \right\| \\ & \lesssim_p s T^{-1/2} (\log(n \vee p \vee T))^{1/2} + \|\tau\|_{\text{MAX}} s^{1/2} n^{-1}. \end{aligned}$$

Since by Assumptions [A.4](#) and [A.5](#), our choice of τ satisfies:

$$\begin{aligned} n^{-1} \|\tau\|_{\text{MAX}} &\lesssim n^{-1} \max_{1 \leq j \leq d} \left\| e_j^\top C_e^\top \widehat{C}_h \right\|_1 \lesssim n^{-1} \|C_e^\top C_h\|_{\text{MAX}} + n^{-1} \left\| C_e^\top (\widehat{C}_h - C_h) \right\|_{\text{MAX}} \\ &\lesssim_p (n^{-1/2} + T^{-1/2}) (\log(n \vee p \vee T))^{1/2}. \end{aligned} \quad (\text{A.26})$$

This concludes the proof of [\(A.24\)](#).

Similarly, to prove [\(A.25\)](#), by [\(A.1\)](#) we have

$$\begin{aligned} &n^{-1/2} \left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}_1])} \left(\widehat{C}_h \check{\lambda}_h + \iota_n \check{\gamma}_0 \right) \right\| \\ &\leq n^{-1/2} \left\| (\iota_n : \widehat{C}_h) (\check{\gamma}_{\widehat{I}_1} - \check{\gamma}_0 : (\check{\lambda}_{\widehat{I}_1} - \check{\lambda}_h)^\top)^\top \right\| \lesssim_p s T^{-1/2} (\log(n \vee p \vee T))^{1/2} + \tau_0 s^{1/2} n^{-1}. \end{aligned}$$

Because we can select τ_0 that satisfies

$$\begin{aligned} n^{-1} \tau_0 &\leq n^{-1} \left\| \lambda_g^\top C_e^\top (\iota_n : \widehat{C}_h) \right\|_1 \leq n^{-1} |\lambda_g^\top C_e^\top \iota_n| + n^{-1} \left\| \lambda_g^\top C_e^\top \widehat{C}_h \right\|_{\text{MAX}} \\ &\lesssim n^{-1} \|C_e \iota_n\|_{\text{MAX}} + \|C_e\|_{\text{MAX}} \left\| \widehat{C}_h - C_h \right\|_{\text{MAX}} + n^{-1} \|C_e^\top C_h\|_{\text{MAX}} \\ &\lesssim_p (n^{-1/2} + T^{-1/2}) (\log(n \vee p \vee T))^{1/2}, \end{aligned}$$

hence it follows that

$$n^{-1/2} \left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}_1])} \left(\widehat{C}_h (\lambda_h + \chi^\top \lambda_g) + \iota_n \gamma_0 \right) \right\| \lesssim_p s (n^{-1/2} + T^{-1/2}) (\log(n \vee p \vee T))^{1/2}.$$

By the triangle inequality and $\mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}_1])} \iota_n = 0$, we have

$$\left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}_1])} \widehat{C}_h \lambda_h \right\| \leq \left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}_1])} \left(\widehat{C}_h (\lambda_h + \chi^\top \lambda_g) + \iota_n \gamma_0 \right) \right\| + \left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}_1])} \widehat{C}_h \chi^\top \right\| \|\lambda_g\|,$$

which, combined with [\(A.24\)](#) and $\|\lambda_g\| \lesssim 1$, lead to the conclusion. \square

Lemma 2. *Under Assumptions [A.1](#), [A.2](#), [A.3](#), [A.4](#), [A.5](#), we have*

$$n^{-1} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \lesssim_p s^2 (n^{-1} + T^{-1}) \log(n \vee p \vee T). \quad (\text{A.27})$$

Proof of Lemma 2. We note by [\(6\)](#) that

$$\widehat{C}_g = \widehat{C}_h \chi^\top + \widehat{C}_g - C_g + \iota_n \xi^\top + (C_h - \widehat{C}_h) \chi^\top + C_e, \quad (\text{A.28})$$

thereby it follows

$$\begin{aligned} n^{-1} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| &\leq n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| + n^{-1} \left\| C_e^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \\ &\quad + n^{-1} \left\| (\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top)^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\|. \end{aligned}$$

On the one hand, by [Lemma 1](#), we have

$$n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \leq n^{-1/2} \left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| n^{-1/2} \left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\|$$

$$\lesssim_p s^2(n^{-1} + T^{-1}) \log(n \vee p \vee T). \quad (\text{A.29})$$

On the other hand, note that

$$\mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h = (\iota_n \gamma_0 + \widehat{C}_h \lambda_h) - (\iota_n : \widehat{C}_h)(\widehat{\gamma}_0 : \widehat{\lambda}_h^\top)^\top = (\iota_n : \widehat{C}_h)(\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top)^\top,$$

where $(\widehat{\gamma}_0 : \widehat{\lambda}_h^\top)^\top = \arg \min_{\gamma, \lambda} \{\iota_n \gamma_0 + \widehat{C}_h \lambda_h - \iota_n \gamma - \widehat{C}_h \lambda : \lambda_j = 0, j \in \widehat{I}^c\}$. By Assumption A.3, we have

$$\begin{aligned} n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| &= n^{-1/2} \left\| (\iota_n : \widehat{C}_h)(\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top)^\top \right\| \\ &\geq \phi_{\min}^{1/2}(s + \widehat{s} + 1) \left[n^{-1} (\iota_n : \widehat{C}_h)^\top (\iota_n : \widehat{C}_h) \right] \left\| (\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top) \right\| \\ &\gtrsim \left\| (\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top) \right\|, \end{aligned}$$

hence it follows from (A.25) that

$$\left\| (\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top) \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}. \quad (\text{A.30})$$

Using this, we have

$$\begin{aligned} n^{-1} \left\| C_e^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| &= n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h)(\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top)^\top \right\| \\ &\lesssim n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h) \right\|_{\text{MAX}} \left\| (\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top)^\top \right\|_1. \end{aligned} \quad (\text{A.31})$$

Using (A.5) and Assumption A.4, it follows that

$$\begin{aligned} n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h) \right\|_{\text{MAX}} &\leq n^{-1} \left\| C_e^\top (\widehat{C}_h - C_h) \right\|_{\text{MAX}} + n^{-1} \|C_e^\top C_h\|_{\text{MAX}} + n^{-1} \|C_e^\top \iota_n\|_{\text{MAX}} \\ &\lesssim \|C_e\|_{\text{MAX}} \left\| \widehat{C}_h - C_h \right\|_{\text{MAX}} + n^{-1} \|C_e^\top C_h\|_{\text{MAX}} + n^{-1} \|C_e^\top \iota_n\|_{\text{MAX}} \\ &\lesssim_p (n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}. \end{aligned} \quad (\text{A.32})$$

Moreover, since by sparsity of λ_h and $\widehat{\lambda}_h$, we have

$$\left\| (\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top)^\top \right\|_1 \leq (s + \widehat{s} + 1)^{1/2} \left\| (\gamma_0 - \widehat{\gamma}_0 : \lambda_h^\top - \widehat{\lambda}_h^\top)^\top \right\|.$$

Combining (A.30), (A.31), and (A.32), we obtain

$$n^{-1} \left\| C_e^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \lesssim_p s^{3/2}(n^{-1} + T^{-1}) \log(n \vee p \vee T). \quad (\text{A.33})$$

Finally, by (A.25) we have

$$\begin{aligned} &n^{-1} \left\| (\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top)^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \\ &\lesssim \left\| \widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right\|_{\text{MAX}} n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \lambda_h \right\| \\ &\lesssim_p s^2(n^{-1/2} T^{-1/2} + T^{-1})(\log(n \vee p \vee T))^{1/2}. \end{aligned}$$

The above estimate, along with (A.33) and (A.29), conclude the proof of (A.27). \square

Lemma 3. Under Assumptions A.1, A.2, A.3, A.4, A.5, we have

$$n^{-1} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}. \quad (\text{A.34})$$

$$n^{-1} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} (\beta_g - \widehat{C}_g \Sigma_z^{-1}) \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}. \quad (\text{A.35})$$

Proof of Lemma 3. (i) By (6), we have

$$\begin{aligned} n^{-1} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\| &\leq n^{-1} \left\| C_e^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\| + n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\| \\ &\quad + n^{-1} \left\| \left((\widehat{C}_g - C_g)^\top + \chi(C_h - \widehat{C}_h)^\top \right) \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\|. \end{aligned}$$

Moreover, by (A.24), we obtain

$$\begin{aligned} n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\| &\leq n^{-1/2} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \right\| n^{-1/2} \|C_h \eta^\top\| \\ &\lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}, \end{aligned} \quad (\text{A.36})$$

where we use the fact that $C_g = C_h \eta^\top + C_z$, and that

$$n^{-1/2} \|C_h \eta^\top\| \lesssim \|C_h \eta^\top\|_{\text{MAX}} \lesssim \|C_g\|_{\text{MAX}} + \|C_z\|_{\text{MAX}} \lesssim 1.$$

In addition, we have

$$n^{-1} \left\| C_e^\top \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\| \leq n^{-1} \|C_e^\top C_h \eta^\top\| + n^{-1} \left\| C_e^\top \mathbb{P}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\|.$$

To bound the first term, we have

$$n^{-1} \|C_e^\top C_h \eta^\top\| \lesssim n^{-1} \|C_e^\top C_h\|_{\text{MAX}} \|\eta\|_\infty \lesssim_p s n^{-1/2} (\log(n \vee p \vee T))^{1/2}.$$

As to the second term, using (A.32) we obtain

$$\begin{aligned} &n^{-1} \left\| C_e^\top \mathbb{P}_{(\iota_n : \widehat{C}_h[\widehat{I}])} C_h \eta^\top \right\| \\ &= n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \left((\iota_n : \widehat{C}_h[\widehat{I}])^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right)^{-1} (\iota_n : \widehat{C}_h[\widehat{I}])^\top C_h \eta^\top \right\| \\ &\leq n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right\| \left\| \left((\iota_n : \widehat{C}_h[\widehat{I}])^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right)^{-1} \right\| \left\| (\iota_n : \widehat{C}_h[\widehat{I}])^\top C_h \eta^\top \right\| \\ &\lesssim (1 + \widehat{s}) \phi_{\min}^{-1} (\widehat{s} + 1) \left[n^{-1} (\iota_n : \widehat{C}_h)^\top (\iota_n : \widehat{C}_h) \right] n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right\|_{\text{MAX}} n^{-1} \left\| (\iota_n : \widehat{C}_h[\widehat{I}])^\top C_h \eta^\top \right\|_{\text{MAX}} \\ &\lesssim_p s (n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}, \end{aligned}$$

where we also use $\|C_h \eta\|_{\text{MAX}} \leq \|C_g\|_{\text{MAX}} + \|C_z\|_{\text{MAX}} \lesssim 1$, and

$$\begin{aligned} n^{-1} \left\| (\iota_n : \widehat{C}_h[\widehat{I}])^\top C_h \eta \right\|_{\text{MAX}} &\leq n^{-1} \left\| (\iota_n : \widehat{C}_h)^\top C_h \eta \right\|_{\text{MAX}} \lesssim \left\| (\iota_n : \widehat{C}_h) \right\|_{\text{MAX}} \|C_h \eta\|_{\text{MAX}} \\ &\lesssim \left(\|(\iota_n : C_h)\|_{\text{MAX}} + \left\| \widehat{C}_h - C_h \right\|_{\text{MAX}} \right) \|C_h \eta\|_{\text{MAX}} \lesssim_p 1. \end{aligned}$$

Therefore, we have

$$n^{-1} \left\| C_e^{\text{TM}}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} C_h \eta^\top \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}. \quad (\text{A.37})$$

Similarly, because we have

$$\begin{aligned} & n^{-1} \left\| \left((\widehat{C}_g - C_g)^\top + \chi(C_h - \widehat{C}_h)^\top \right) C_h \eta^\top \right\| \\ & \lesssim \left\| (\widehat{C}_g - C_g)^\top + \chi(C_h - \widehat{C}_h)^\top \right\|_{\text{MAX}} \|C_h \eta^\top\|_{\text{MAX}} \lesssim_p sT^{-1/2}(\log(n \vee p \vee T))^{1/2}. \\ & n^{-1} \left\| \left((\widehat{C}_g - C_g)^\top + \chi(C_h - \widehat{C}_h)^\top \right) (\iota_n : \widehat{C}_h[\widehat{\Gamma}]) \right\|_{\text{MAX}} \\ & \leq K \left\| (\widehat{C}_g - C_g)^\top + \chi(C_h - \widehat{C}_h)^\top \right\|_{\text{MAX}} \left\| (\iota_n : \widehat{C}_h[\widehat{\Gamma}]) \right\|_{\text{MAX}} \lesssim_p sT^{-1/2}(\log(n \vee p \vee T))^{1/2}, \end{aligned}$$

it follows that

$$n^{-1} \left\| \left((\widehat{C}_g - C_g)^\top + \chi(C_h - \widehat{C}_h)^\top \right) \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} C_h \eta^\top \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2},$$

which, along with (A.36) and (A.37), establish the first claim.

(ii) Next, by (5) we have

$$\widehat{C}_g = \widehat{C}_h \eta^\top + \widehat{C}_z.$$

And recall that $\beta_g = C_z \Sigma_z^{-1}$, so we have

$$\begin{aligned} & n^{-1} \left\| \widehat{C}_g^{\text{TM}}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} (\beta_g - \widehat{C}_g \Sigma_z^{-1}) \right\| \\ & \leq n^{-1} \left\| \widehat{C}_g^{\text{TM}}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} (C_z - \widehat{C}_z) \Sigma_z^{-1} \right\| + n^{-1} \left\| \widehat{C}_g^{\text{TM}}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} (\widehat{C}_h - C_h) \eta^\top \Sigma_z^{-1} \right\| \\ & \quad + n^{-1} \left\| \widehat{C}_g^{\text{TM}}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} C_h \eta^\top \Sigma_z^{-1} \right\|. \end{aligned}$$

Using Assumption A.4 and $\left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} \right\| \leq 1$, we have

$$\begin{aligned} & n^{-1} \left\| \widehat{C}_g^{\text{TM}}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} (C_z - \widehat{C}_z) \Sigma_z^{-1} \right\| \\ & \lesssim \left\| \widehat{C}_g \right\|_{\text{MAX}} \left\| C_z - \widehat{C}_z \right\|_{\text{MAX}} \left\| \Sigma_z^{-1} \right\| \lesssim_p T^{-1/2}(\log(n \vee p \vee T))^{1/2}, \end{aligned} \quad (\text{A.38})$$

where we also use the fact that

$$\left\| \Sigma_z^{-1} \right\| \leq \lambda_{\min}^{-1}(\Sigma_z) \lesssim 1, \quad \left\| \widehat{C}_g \right\|_{\text{MAX}} \leq \left\| \widehat{C}_g - C_g \right\|_{\text{MAX}} + \|C_g\|_{\text{MAX}} \lesssim 1.$$

Similarly, we obtain

$$\begin{aligned} n^{-1} \left\| \widehat{C}_g^{\text{TM}}_{(\iota_n: \widehat{C}_h[\widehat{\Gamma}])} (\widehat{C}_h - C_h) \eta^\top \Sigma_z^{-1} \right\| & \lesssim \left\| \widehat{C}_g \right\|_{\text{MAX}} \left\| \widehat{C}_h - C_h \right\|_{\text{MAX}} \|\eta\|_\infty \left\| \Sigma_z^{-1} \right\| \\ & \lesssim_p sT^{-1/2}(\log(n \vee p \vee T))^{1/2}. \end{aligned} \quad (\text{A.39})$$

Combining (A.38), (A.39), and (A.34) concludes the proof. \square

Lemma 4. Under Assumptions A.1, A.2, A.3, A.4, A.5, we have

$$\begin{aligned} & n^{-1} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])}(\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \lesssim_p s^2 (n^{-1/2} T^{-1/2} + T^{-1}) \log(n \vee p \vee T). \end{aligned} \quad (\text{A.40})$$

Proof of Lemma 4. From (A.24) and Assumption A.4, it follows that

$$\begin{aligned} & n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])}(\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \leq n^{-1/2} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \right\| \|\beta_g \eta + \beta_h\|_\infty \left(\|T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h)\|_{\text{MAX}} + \|\bar{h}\|_{\text{MAX}} \right) \\ & \lesssim_p s^2 (n^{-1/2} T^{-1/2} + T^{-1}) \log(n \vee p \vee T). \end{aligned} \quad (\text{A.41})$$

Next, by triangle inequality, we have

$$\begin{aligned} & n^{-1} \left\| C_e^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])}(\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{Z}^\top \lambda_g + (T^{-1} \bar{H} \bar{H}^\top - \Sigma_h) (\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \leq n^{-1} \left\| C_e^\top (\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \quad + n^{-1} \left\| C_e^\top \mathbb{P}_{(\iota_n: \widehat{C}_h[\widehat{I}])}(\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\|. \end{aligned}$$

For the first term, by Assumption A.5 we have

$$\begin{aligned} & n^{-1} \left\| C_e^\top (\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \leq n^{-1} \|C_e^\top (\beta_g \eta + \beta_h)\|_\infty \left\| (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\|_{\text{MAX}} \\ & \lesssim_p s n^{-1/2} T^{-1/2} (\log(n \vee p \vee T))^{1/2}. \end{aligned}$$

For the second term, we use Assumptions A.1, A.3, A.4, and (A.32),

$$\begin{aligned} & n^{-1} \left\| C_e^\top \mathbb{P}_{(\iota_n: \widehat{C}_h[\widehat{I}])}(\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \lesssim (1 + \widehat{s}) \phi_{\min}^{-1}(\widehat{s} + 1) \left[n^{-1} (\iota_n : \widehat{C}_h)^\top (\iota_n : \widehat{C}_h) \right] n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right\|_{\text{MAX}} \\ & \quad \times \left\| (\iota_n : \widehat{C}_h[\widehat{I}])^\top \right\|_{\text{MAX}} \|\beta_g \eta + \beta_h\|_\infty \left\| T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h} \right\|_{\text{MAX}} \\ & \lesssim_p s^2 (n^{-1/2} T^{-1/2} + T^{-1}) \log(n \vee p \vee T). \end{aligned}$$

Finally, by Assumptions A.1 and A.4, we have

$$\begin{aligned} & n^{-1} \left\| (\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top)^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])}(\beta_g \eta + \beta_h) (T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h}) \right\| \\ & \lesssim \left\| (\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top)^\top \right\|_{\text{MAX}} \|\beta_g \eta + \beta_h\|_\infty \left\| T^{-1} \bar{H} \bar{V}^\top \lambda - \Sigma_h(\eta^\top \lambda_g + \lambda_h) - \bar{h} \right\|_{\text{MAX}} \\ & \lesssim_p s^2 T^{-1} \log(n \vee p \vee T). \end{aligned}$$

The conclusion then follows from (A.28). \square

Lemma 5. Under Assumptions A.1, A.2, A.3, A.4, we have

$$n^{-1} \left\| \widehat{C}_g^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \lesssim_p s(n^{-1/2} T^{-1/2} + T^{-1}) \log(n \vee p \vee T). \quad (\text{A.42})$$

Proof of Lemma 5. Note that by (A.24), we have

$$\begin{aligned} n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| &\leq n^{-1/2} \left\| \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| n^{-1/2} \left\| \bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda \right\| \\ &\lesssim_p s(n^{-1/2} T^{-1/2} + T^{-1}) \log(n \vee p \vee T), \end{aligned}$$

where we use the following estimates as a result of Assumptions A.1 and A.4:

$$\begin{aligned} n^{-1/2} \|\bar{u}\| &\lesssim \|\bar{u}\|_{\text{MAX}} \lesssim_p T^{-1/2} (\log n \vee p \vee T)^{1/2}, \\ n^{-1/2} \|T^{-1} \bar{U} \bar{V}^\top \lambda\| &\lesssim \|T^{-1} \bar{U} \bar{M}^\top \gamma_0\|_{\text{MAX}} \lesssim_p T^{-1/2} (\log(n \vee p \vee T))^{1/2}. \end{aligned}$$

Moreover, by triangle inequality, we have

$$\begin{aligned} &n^{-1} \left\| C_e^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \\ &\leq n^{-1} \left\| C_e^\top (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| + n^{-1} \left\| C_e^\top \mathbb{P}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \end{aligned}$$

For the first term, we have

$$n^{-1} \left\| C_e^\top (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \leq n^{-1} \left\| C_e^\top \bar{u} \right\| + T^{-1} n^{-1} \left\| C_e^\top \bar{U} \bar{V}^\top \lambda \right\| \lesssim_p s n^{-1/2} T^{-1/2}.$$

As to the second term, using Assumption A.3 and (A.32) we have

$$\begin{aligned} &n^{-1} \left\| C_e^\top \mathbb{P}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \\ &= n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \left((\iota_n : \widehat{C}_h[\widehat{I}])^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right)^{-1} (\iota_n : \widehat{C}_h[\widehat{I}])^\top (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \\ &\lesssim s n^{-1} \left\| C_e^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right\|_{\text{MAX}} n^{-1} \left\| (\iota_n : \widehat{C}_h[\widehat{I}])^\top (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\|_{\text{MAX}} \\ &\lesssim_p s (n^{-1/2} T^{-1/2} + T^{-1}) \log(n \vee p \vee T), \end{aligned}$$

where we also use the following

$$\begin{aligned} n^{-1} \left\| (\iota_n : \widehat{C}_h)^\top (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\|_{\text{MAX}} &\leq \left(\left\| \widehat{C}_h - C_h \right\|_{\text{MAX}} + \|(\iota_n : C_h)\|_{\text{MAX}} \right) \left\| \bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda \right\|_{\text{MAX}} \\ &\lesssim_p T^{-1/2} (\log(n \vee p \vee T))^{1/2}. \end{aligned}$$

Finally, we note that

$$\begin{aligned} &n^{-1} \left\| \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right)^\top \mathbb{M}_{(\iota_n: \widehat{C}_h[\widehat{I}])} (\bar{u} - T^{-1} \bar{U} \bar{V}^\top \lambda) \right\| \\ &\lesssim \left\| \widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right\|_{\text{MAX}} \left\| \bar{u} - T^{-1} \bar{U} \bar{M}^\top \right\|_{\text{MAX}} \lesssim_p s T^{-1} \log(n \vee p \vee T). \end{aligned}$$

This concludes the proof. \square

Lemma 6. Under Assumptions A.1, A.2, A.3, A.4, A.5, we have

$$\left\| n \left(\widehat{C}_g^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_g \right)^{-1} \right\| \lesssim_p 1.$$

Proof of Lemma 6. Note that by (A.28), we have

$$\begin{aligned} & \widehat{C}_g^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_g \\ &= C_e^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} C_e + C_e^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top + \chi \widehat{C}_h^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} C_e + \chi \widehat{C}_h^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \\ & \quad + C_e^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) + \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right)^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} C_e \\ & \quad + \chi \widehat{C}_h^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) + \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right)^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \\ & \quad + \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right)^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) \end{aligned}$$

There are 9 terms in total on the right-hand side. By (A.24), we have

$$\begin{aligned} n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} C_e \right\| &= n^{-1} \left\| C_e^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| \lesssim \|C_e\|_{\text{MAX}} n^{-1/2} \left\| \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| \\ &\lesssim_p s (n^{-1/2} + T^{-1/2}) (\log(n \vee p \vee T))^{1/2}, \\ n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| &\leq n^{-1} \left\| \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\|^2 \lesssim_p s^2 (n^{-1} + T^{-1}) \log(n \vee p \vee T). \end{aligned}$$

Also, we have

$$\begin{aligned} n^{-1} \left\| C_e^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) \right\| &= n^{-1} \left\| \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} C_e \right\| \\ &\lesssim \|C_e\|_{\text{MAX}} \left\| \widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right\|_{\text{MAX}} \lesssim_p s T^{-1/2} (\log(n \vee p \vee T))^{1/2}, \\ n^{-1} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) \right\| &= n^{-1} \left\| \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_h \chi^\top \right\| \\ &\lesssim n^{-1/2} \left\| \chi \widehat{C}_h^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \right\| \left\| \widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right\|_{\text{MAX}} \\ &\lesssim_p s^2 (n^{-1/2} T^{-1/2} + T^{-1}) \log(n \vee p \vee T), \\ n^{-1} \left\| \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right)^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \left(\widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right) \right\| & \\ &\lesssim \left\| \widehat{C}_g - C_g + (C_h - \widehat{C}_h) \chi^\top \right\|_{\text{MAX}}^2 \lesssim_p s^2 T^{-1} \log(n \vee p \vee T). \end{aligned}$$

Finally, by (A.32) and Assumptions A.2 and A.3, we have

$$\begin{aligned} n^{-1} \left\| C_e^\top \mathbb{P}_{(\ell_n: \widehat{C}_h[\widehat{I}])} C_e \right\| &= n^{-1} \left\| C_e^\top (\ell_n : \widehat{C}_h[\widehat{I}]) \left((\ell_n : \widehat{C}_h[\widehat{I}])^\top (\ell_n : \widehat{C}_h[\widehat{I}]) \right)^{-1} (\ell_n : \widehat{C}_h[\widehat{I}])^\top C_e \right\| \\ &\lesssim s n^{-2} \left\| C_e^\top (\ell_n : \widehat{C}_h[\widehat{I}]) \right\|_{\text{MAX}}^2 \lesssim_p s (n^{-1} + T^{-1}) \log(n \vee p \vee T). \end{aligned}$$

Hence, we obtain

$$n^{-1} \widehat{C}_g^\top \mathbb{M}_{(\ell_n: \widehat{C}_h[\widehat{I}])} \widehat{C}_g = n^{-1} C_e^\top C_e + o_p(1).$$

The conclusion follows from (A.5) and Weyl inequalities. \square

Lemma 7. *Under Assumptions A.1, A.2, A.3, A.4, A.5, A.6, we have*

$$\left\| (\widehat{\gamma}_0 : \widehat{\lambda}_h^\top) - (\gamma_0 : \lambda_h^\top) \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}.$$

Proof. It follows from (9) that

$$(\widehat{\gamma}_0 : \widehat{\lambda}_h[\widehat{I}]^\top)^\top = \left((\iota_n : \widehat{C}_h[\widehat{I}])^\top (\iota_n : \widehat{C}_h[\widehat{I}]) \right)^{-1} (\iota_n : \widehat{C}_h[\widehat{I}])^\top (\bar{r} - \widehat{C}_g \widehat{\lambda}_g),$$

which implies that

$$\left\| (\widehat{\gamma}_0 : \widehat{\lambda}_h^\top)^\top - (\gamma_0 : \lambda_h^\top)^\top \right\| \leq \left\| (\widetilde{\gamma}_0 : \widetilde{\lambda}_h^\top)^\top - (\check{\gamma}_0 : \check{\lambda}_h^\top)^\top \right\| + \left\| (\widetilde{\xi} : \widetilde{\chi})^\top \widehat{\lambda}_g - (\xi : \chi)^\top \lambda_g \right\|,$$

where

$$\begin{aligned} (\widetilde{\gamma}_0 : \widetilde{\lambda}_h^\top)^\top &= \arg \min_{\gamma, \lambda} \left\{ \left\| \bar{r} - \iota_n \gamma - \widehat{C}_h \lambda \right\| : \lambda_j = 0, \quad j \notin \widehat{I} \right\}, \\ (\widetilde{\xi}_j : \widetilde{\chi}_{j,\cdot})^\top &= \arg \min_{\xi_j, \chi_{j,\cdot}} \left\{ \left\| \widehat{C}_{g,\cdot,j} - \iota_n \xi_j - \widehat{C}_h \chi_{j,\cdot} \right\| : \chi_{j,k} = 0, \quad k \notin \widehat{I} \right\}, \quad j = 1, 2, \dots, d. \end{aligned}$$

Moreover, because

$$\mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}])} \bar{r} = \iota_n \check{\gamma}_0 + \widehat{C}_h \check{\lambda}_h - \iota_n \widetilde{\gamma}_0 - \widehat{C}_h \widetilde{\lambda}_h + (C_h - \widehat{C}_h) \check{\lambda}_h + C_e \lambda_g + \beta_g \bar{g} + \beta_h \bar{h} + \bar{u}$$

we obtain, using $\widehat{I}_1 \subseteq \widehat{I}$, (A.1), (A.5), (A.20) - (A.23), (A.26),

$$\begin{aligned} & n^{-1/2} \left\| (\iota_n : \widehat{C}_h) \left(\widetilde{\gamma}_0 - \check{\gamma}_0 : (\widetilde{\lambda}_h - \check{\lambda}_h)^\top \right)^\top \right\| \\ & \leq n^{-1/2} \left\| \mathbb{M}_{(\iota_n : \widehat{C}_h[\widehat{I}_1])} \bar{r} \right\| + n^{-1/2} \left\| (C_h - \widehat{C}_h) \check{\lambda}_h + C_e \lambda_g + \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} \right\| \\ & \leq n^{-1/2} \left\| (\iota_n : \widehat{C}_h) \left(\widetilde{\gamma}_{\widehat{I}_1} - \check{\gamma}_0 : (\widetilde{\lambda}_{\widehat{I}_1} - \check{\lambda}_h)^\top \right)^\top \right\| + 2n^{-1/2} \left\| (C_h - \widehat{C}_h) \check{\lambda}_h + C_e \lambda_g + \beta_g \bar{g} + \beta_h \bar{h} + \bar{u} \right\| \\ & \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}. \end{aligned}$$

Since we have

$$\begin{aligned} & n^{-1/2} \left\| (\iota_n : \widehat{C}_h) \left(\widetilde{\gamma}_0 - \check{\gamma}_0 : (\widetilde{\lambda}_h - \check{\lambda}_h)^\top \right)^\top \right\| \\ & \geq \phi_{\min}^{1/2}(1 + \widehat{s}) \left[n^{-1} (\iota_n : \widehat{C}_h)^\top (\iota_n : \widehat{C}_h) \right] \left\| \left(\widetilde{\gamma}_0 - \check{\gamma}_0 : (\widetilde{\lambda}_h - \check{\lambda}_h)^\top \right) \right\|, \end{aligned}$$

it follows that

$$\left\| \left(\widetilde{\gamma}_0 - \check{\gamma}_0 : (\widetilde{\lambda}_h - \check{\lambda}_h)^\top \right) \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}.$$

Similarly, we can obtain

$$\left\| \left(\widetilde{\xi} - \xi : \widetilde{\chi} - \chi \right) \right\| \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}.$$

Therefore, using this, as well as Assumption A.1 and Theorem 1, we obtain

$$\begin{aligned} \left\| (\widetilde{\xi} : \widetilde{\chi})^\top \widehat{\lambda}_g - (\xi : \chi)^\top \lambda_g \right\| & \leq \left\| (\widetilde{\xi} - \xi : \widetilde{\chi} - \chi) \right\| \left\| \widehat{\lambda}_g \right\| + \left\| (\xi : \chi) \right\| \left\| \widehat{\lambda}_g - \lambda_g \right\| \\ & \lesssim_p s(n^{-1/2} + T^{-1/2})(\log(n \vee p \vee T))^{1/2}. \end{aligned}$$

This concludes the proof. \square